1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 03 (Class XII, Semester - 1)
Module Name/Title	Application of Derivatives – Part 4
Module Id	lemh_10604
Pre-requisites	Basic knowledge about Slopes and Equations of Tangents and Normals
Objectives	After going through this lesson, the learners will be able to understand the following: • Maximum and Minimum • Local Maxima and Local Minima • First Derivative Test for Local Maxima and Local Minima • Second Derivative Test for Local Maxima and Local Minima
Keywords	Maximum and Minimum, Local Maxima, Local Minima, First Derivative Test, Second Derivative Test, Closed Interval

2. Development Team

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1. Introduction:

In previous modules, the use of derivatives has been discussed to determine the rate of change of various quantities and to find the intervals in which a given function increases or decreases. We have also learnt to use differentiation to find the equations of tangents and normals to a given curve. In this module, the concept of derivatives will be used to find the maximum and minimum values of various differentiable functions in their domains. We will define points of local maxima and local minima and obtain them using differentiation. We will also discuss some problems related with maximum and minimum.

In our day to day life there are many situations where we might encounter maxima and minima. Let us consider certain examples;

Suppose a fire ball is shot form a cannon then we may be interested to know the maximum height to which the fire ball rises before falling on the ground.

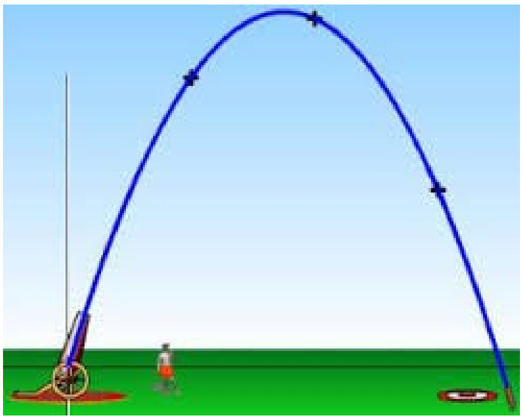

Fig. 1

A ball, thrown into the air, rises to a maximum height, falls and then rise and fall continues till the stops.. But the maximum height to which it rises every time does not remain same.

Fig. 2

Suppose a helicopter of enemy is flying in the sky then a soldier will shoot the helicopter when it is nearest to him. Hence we are
interested to know the nearest distance.

An engineer working on space shuttle will compute maximum pressure acting on the shuttle at a given altitude. The absolute maximum of the pressure acting on the shuttle will guide him to design the shuttle to sustain that pressure.

When dealing with costs, we would like to minimize it and with profit we always want to maximize. So we need to calculate both maximum and minimum values.

There are plenty of other examples, in fact maximum and minimum values are needed in almost all the situations in our daily life. In order to tackle such problems, we first define a function to represent the problem and then find maximum or minimum values of the function.

Let us have a look on the graph of a function given below,

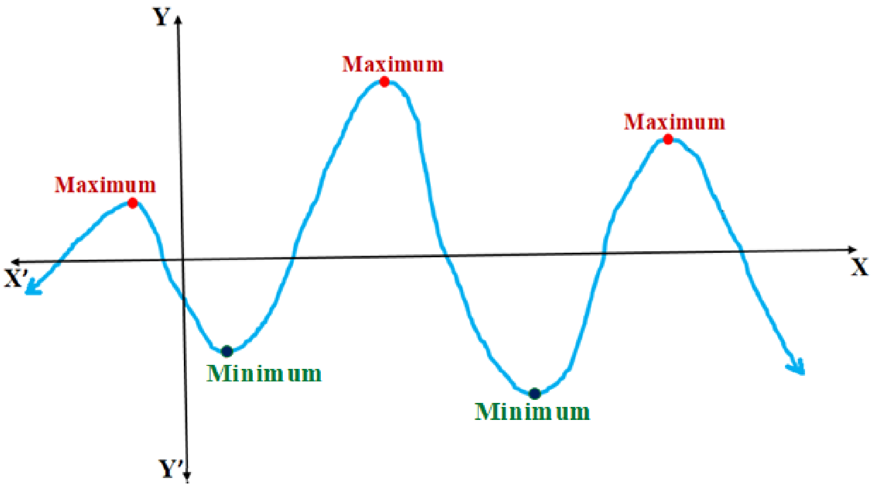

Fig. 3

A maximum is a highest point and a minimum is a lowest point in the graph in its neighborhood. Red dots in the above graph are the points of maximum and the black dots are the points of minimum.

2. Maximum and Minimum Values of a Function in its Domain:

Let f be a function defined on an interval I. Then

(a) f is said to have a maximum value in I, if there exists a point c in I such that $f(c) \geq f(x)$, for all $x \in$ I.

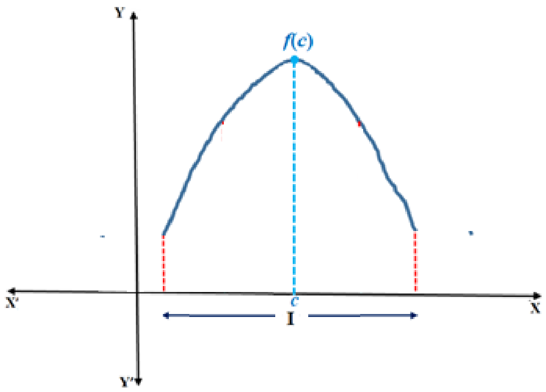

Fig. 4

The number $f(c)$ is called the maximum value of f in I and the point c is called a point of maximum value of f in I.

(b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) \leq f(x)$, for all $x \in$ I.

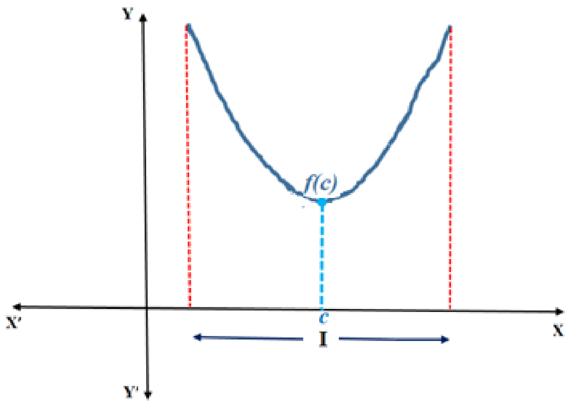

Fig. 5

The number $f(c)$, in this case, is called the minimum value of f in I and the point c, is called a point of minimum value of f in I.

(c) f is said to have an extreme value in I if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I. The number $f(c)$ is called an extreme value of f in I and the point c is called an extreme point.

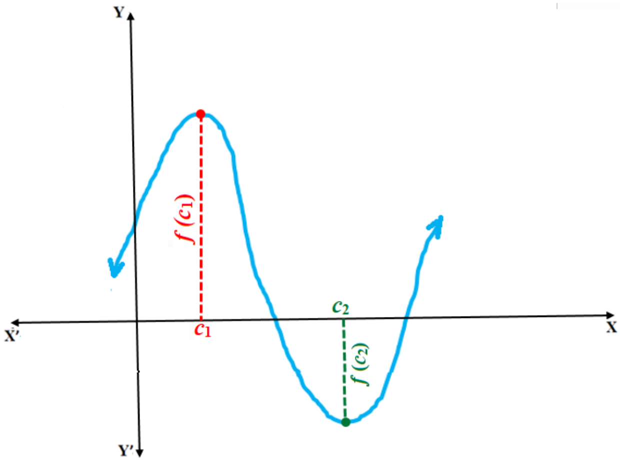

Fig. 6

In Fig. 6, c_1 is the point of maximum value of f and the maximum value of the function is $f(c_1)$ and c_2 is the point of minimum value of f and the minimum value of the function is $f(c_2)$. Thus c_1 and c_2 are the extreme points, $f(c_1)$ and $f(c_2)$ are the extreme values of the function.

Example 1 :

Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = 9x^2 + 12x + 2$

(ii) $f(x) = -|x + 1| + 3$

(iii) $f(x) = x^3 + 1$

Solution :

(i) The given function is

$f(x) = 9x^2 + 12x + 2$

$= (3x + 2)^2 - 2$

We know that, $(3x + 2)^2 \geq 0$, for all $x \in R$

Hence, $f(x) \geq -2$, for all $x \in R$,

Therefore, the minimum value of the function is -2 and it attains its minimum value at $x = -\frac{2}{3}$ . .

Since the value of $f(x)$ goes on increasing, the function does not have any maximum value, see Fig.7.

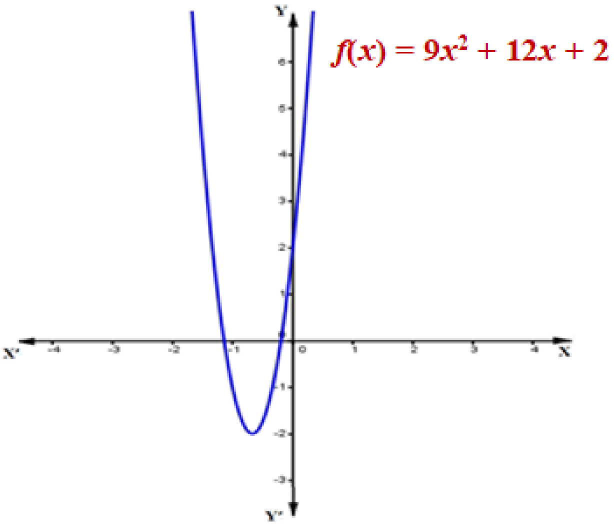

Fig. 7

(ii) The second function is,

$f(x) = -|x + 1| + 3$

We know that, $|x + 1| \geq 0$,

So, $-|x + 1| \leq 0$, for all $x \in R$

Hence, $f(x) \leq 3$, for all $x \in R$,

Hence the maximum value of the function is 3 and it attains its maximum value at $x = -1$.

Since the value of $f(x)$ can be made as small as we please by changing the values of x, so the function does not have any minimum value, as is shown in the Fig. 8.

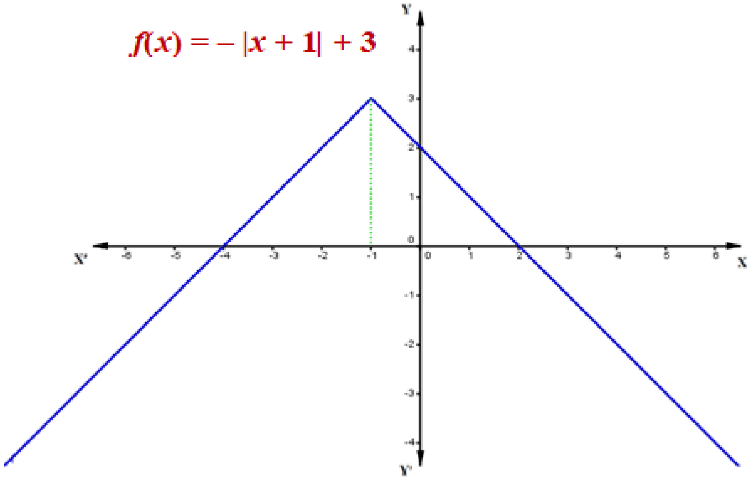

Fig. 8

(iii) The next function is; $f(x) = x^3 + 1$

Look below at the graph of the function,

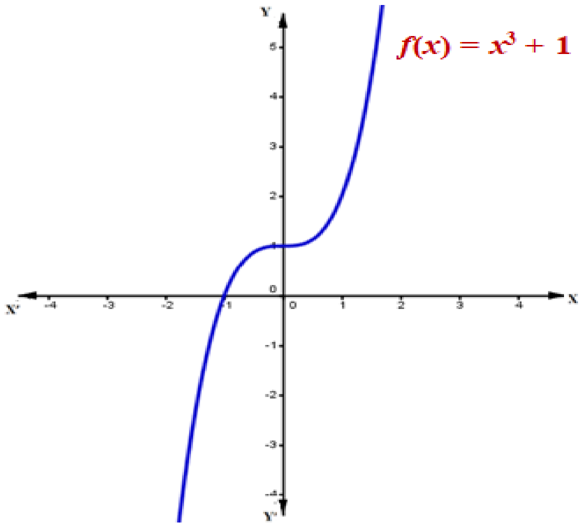

Fig. 9

We observe that the values of $f(x)$ increase as we increase the values of x and the value of $f(x)$ can be made as large as we please by giving large values to x, so $f(x)$ does not have any maximum value. Similarly, $f(x)$ can be made as small as we please by giving small values to x, so $f(x)$ does not have any minimum value also.

But if we restrict the domain of f to any closed interval as [-2, 2], then function will have both maximum and minimum values.

Maximum value; $f(2) = 9$ and Minimum value; $f(-2) = -7$

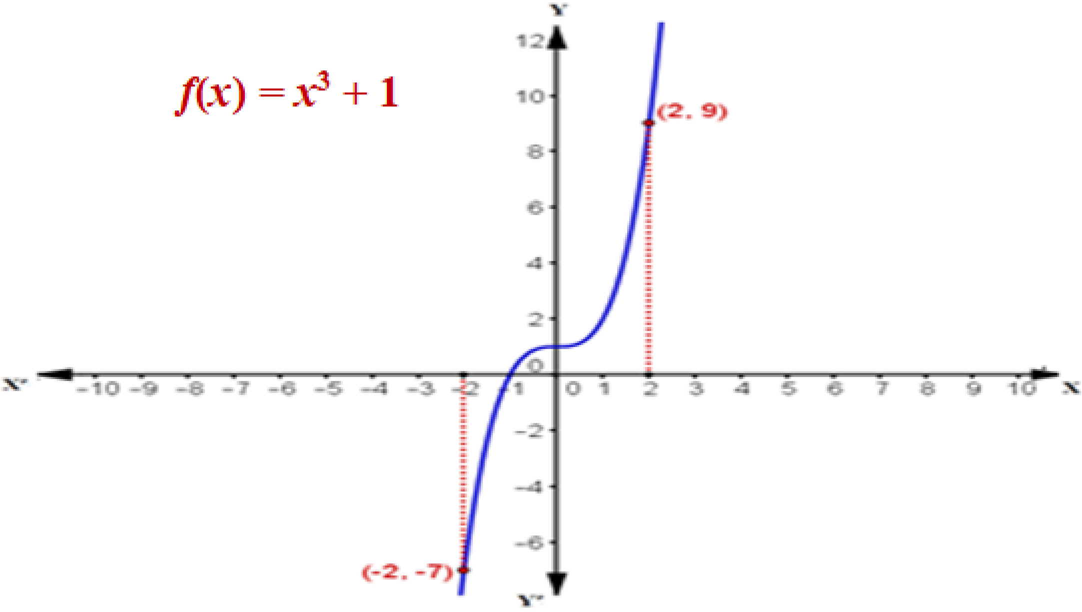

Fig. 10

Every continuous function on a closed interval has a maximum and a minimum value.

Note:

It follows from the above discussion that a function defined on an interval I,

(i) May attain both maximum and minimum values at some point in I.

(ii) May attain maximum value at a point in I but not the minimum value at any point in I.

(iii) May attain minimum value at a point in I but not the maximum value at any point in I.

(iv) May not attain both maximum and minimum values at any point in I.

3. Local Maxima and Local Minima:

Study carefully the graph of a function given below. Observe that at points A, B, C, D and E on the graph, the function changes its nature from increasing to decreasing or decreasing to increasing. These points are called turning points of the given function. See at turning points, the graph has either a little hill or a little valley. Roughly speaking, the function has maximum value in some neighbourhood (interval) of each of the points A, C and E which are at the top of their respective hills.

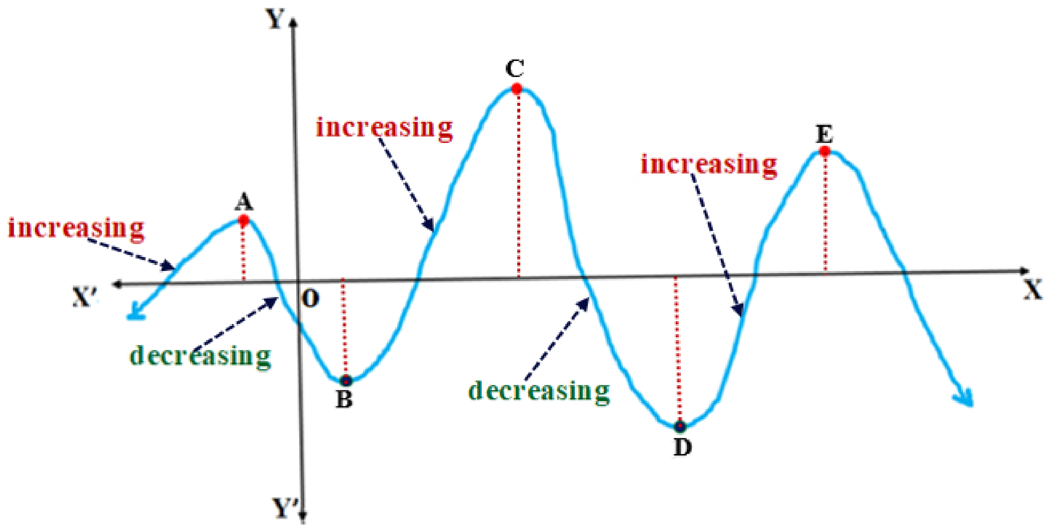

Fig. 11

Similarly, the function has minimum value in some neighbourhood of points B and D which are at the bottom of their respective valleys. Hence, the points A, C and E may be regarded as points of relative maximum value (or local maximum value) and points B and D may be regarded as points of relative minimum value (or local minimum value) for the function. The local maximum value and local minimum value of the function are also known as local maxima and local minima of the function, respectively.

Definition:

Let, f be a real valued function and c be an interior point in the domain of f. Then,

(a) c is called a point of local maxima, if there exists a real number h, $h > 0$, such that

$$f(c) > f(x), \quad \text{for all } x \in (c - h, c + h)$$

The value $f(c)$ is called the local maximum value of f.

(b) c is called a point of local minima, if there exists a real number h, $h > 0$, such that

$$f(c) < f(x), \quad \text{for all } x \in (c - h, c + h)$$

The value $f(c)$ is called the local minimum value of f.

Geometrical Interpretation:

Geometrically, the above definition states that if, $x = c$ is a point of local maxima of function f, then the graph of f around c will be as shown in the Fig. 12.

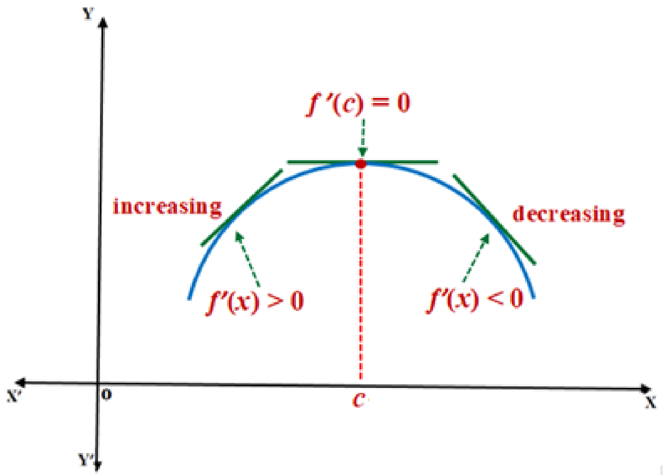

Fig. 12

From the graph you will note that, f is increasing (i.e., $f'(x) > 0$) in the interval $(c - h, c)$ and decreasing (i.e., $f'(x) < 0$) in the interval $(c, c + h)$. This suggests that $f'(c)$ must be zero.

Similarly, if c is a point of local minima of f, then graph of the function will be as shown below, Fig. 13

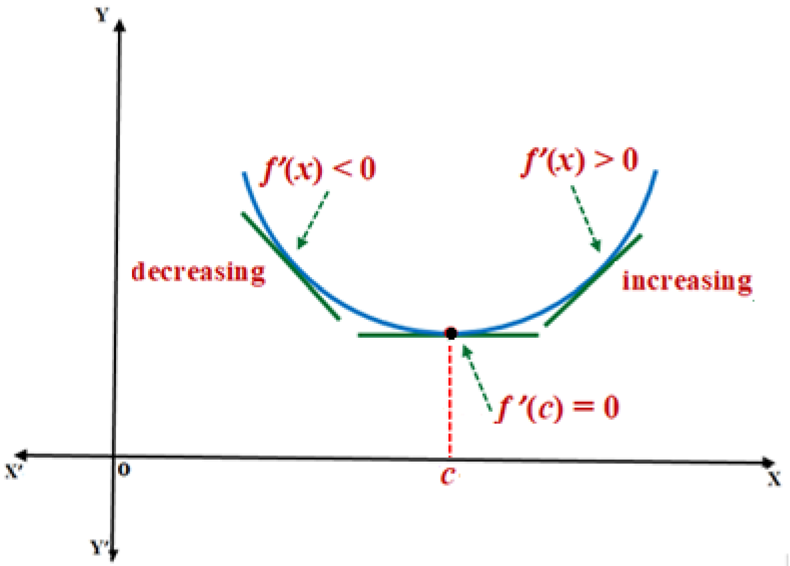

Fig. 13

See in this graph, f is decreasing (i.e., $f'(x) < 0$) in the interval $(c - h, c)$ and increasing (i.e., $f'(x) > 0$) in the interval $(c, c + h)$. This suggests that $f'(c)$ must be zero.

On the basis of the above discussion we may state following theorem (without proof).

Theorem:

Let f be a function defined on an open interval I.

Suppose, c ∈ I, be any point, then, if f has a local maxima or a local minima at $x = c$, then either $f'(c) = 0$ or f is not differentiable at c.

Note:

The converse of above theorem need not be true, a point at which the derivative vanishes need not be a point of local maxima or local minima.

For example,

If, $f(x) = x^3$, then $f'(x) = 3x^2$ and $f'(0) = 0$.

But, $x = 0$ is neither a point of local maxima nor a point of local minima. See the graph below, before $x = 0$ the function is increasing and also after $x = 0$ the function is increasing.

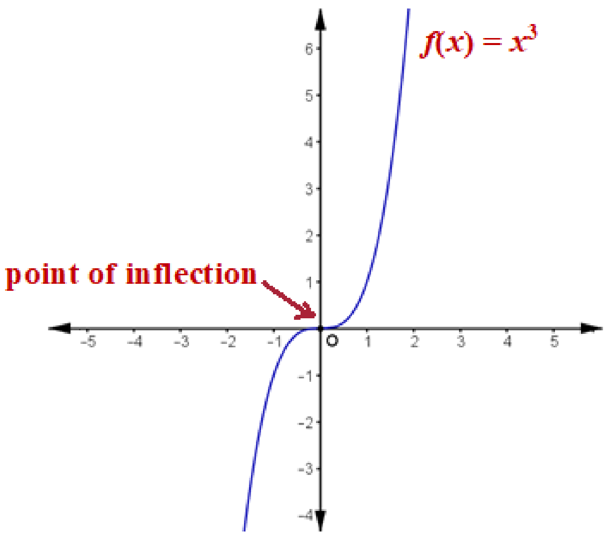

Fig. 14

Here, $x = 0$ is neither a point of local maxima nor a point of local minima, such a point is known as point of inflection.

Note:

If we have a point c in the domain of a function f at which, either $f'(c) = 0$ or f is not differentiable, then such a point 'c' is called a critical point of the function f.

On the basis of the above discussion we shall now learn working rule for finding points of local maxima or points of local minima using only the first order derivatives.

4. Local Maxima and Local Minima (First Derivative Test):

Let f be a function defined on an open interval I and let f be continuous at a critical point c in I. Then,

(i) If $f'(x)$ changes sign from positive to negative as x increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to c and to the left of c, and $f'(x) < 0$ at every point sufficiently close to c and to the right of c, then c is a point of local maxima.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c, i.e., if $f'(x) < 0$ at every point sufficiently close to c and to the left of c, and $f'(x) > 0$ at every point sufficiently close to c and to the right of c, then c is a point of local minima.

Compare, two figures given below to understand the two rules given above;

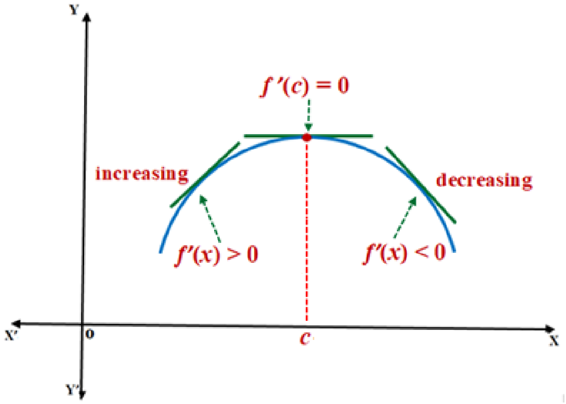

Fig. 15 (i)

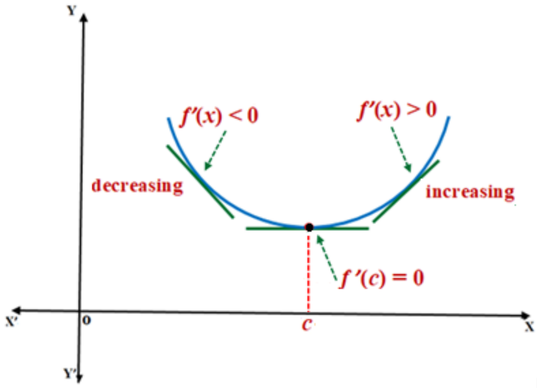

Fig. 15 (ii)

(iii) If $f'(c) = 0$ and $f'(x)$ does not change sign as x increases or decreases through c, then c is neither a point of local maxima nor a point of local minima. And such a point is called point of inflection. See the Fig. 16, below;

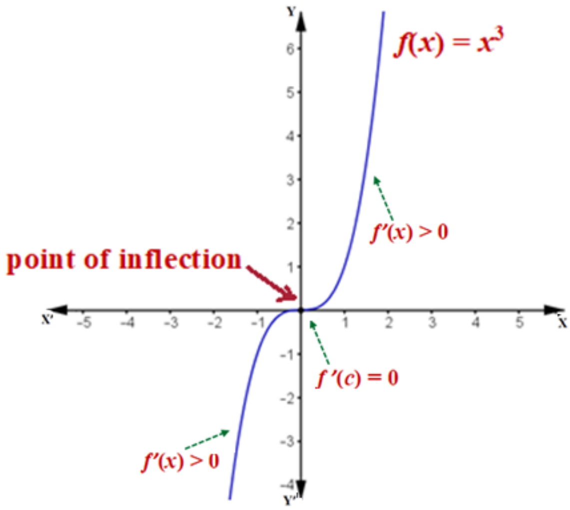

Fig. 16

Example 2 :

Find all the points of local maxima and local minima of the function f given by,

$$f(x) = x^3 - 3x + 7$$

Solution :

The given function is;

$$f(x) = x^3 - 3x + 7 \quad \ldots\ldots\ldots\ldots(i)$$

Hence, $\quad f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$

For local maxima and local minima, we must have,

$$f'(x) = 0,$$

$$\Rightarrow \quad x = 1, -1$$

Hence, $x = 1$ and $x = -1$ are the critical points of the given function. Thus, $x = \pm 1$ are the points which could possibly be the points of local maxima or local minima of the function.

Let us first examine the point $x = 1$.

For the values of x close to 1 and to the left of 1, we have $f'(x) < 0$ and for the values of x close to 1 and to the right of 1 we have $f'(x) > 0$. Therefore, by first derivative test, $x = 1$ is a point of local minima.

Local minimum value = $f(1) = 5$.

In the case of $x = -1$, we observe that $f'(x) > 0$, for values of x close to -1 and to the left of -1 and $f'(x) < 0$, for values of x close to -1 and to the right of -1. Therefore, by first derivative test, $x = -1$ is a point of local maxima.

Local maximum value is = $f(-1) = 9$.

See below the graph of the function to understand the explantion.

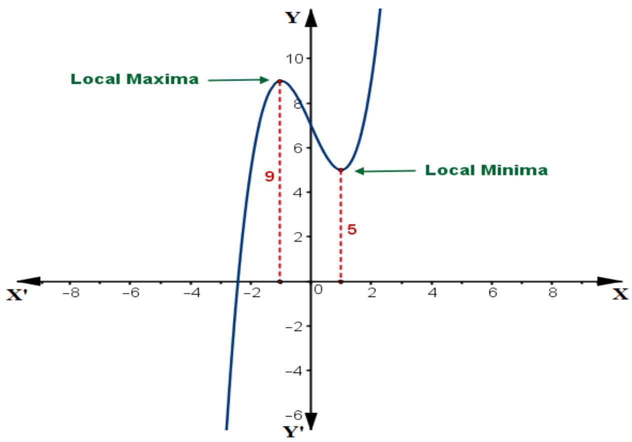

Fig. 17

Example 3 :

Find all the points of local maxima and local minima of the function f given by,

$$f(x) = x^3 - 6x^2 + 12x - 8$$

Solution :

The given function is,

$$f(x) = x^3 - 6x^2 + 12x - 8$$

So, $$f'(x) = 3x^2 - 12x + 12$$

$$= 3(x-2)^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

For local maxima and local minima and we must have,

$$f'(x) = 0, \Rightarrow \quad x = 2,$$

hence, $x = 2$ is the critical point of the given function. Let us now examine the point for local maxima or local minima.

From (i), $f'(x) > 0$, for all values of x to the left of 2 and close to 2,

similarly, $f'(x) > 0$, for values of x to the right of 2 and close to 2.

Thus, $f'(x)$ does not change sign as x passes through $x = 2$.

Therefore, by first derivative test, the point $x = 2$ is neither a point of local maxima nor a point of local minima.

Hence, $x = 2$ is a point of inflection.

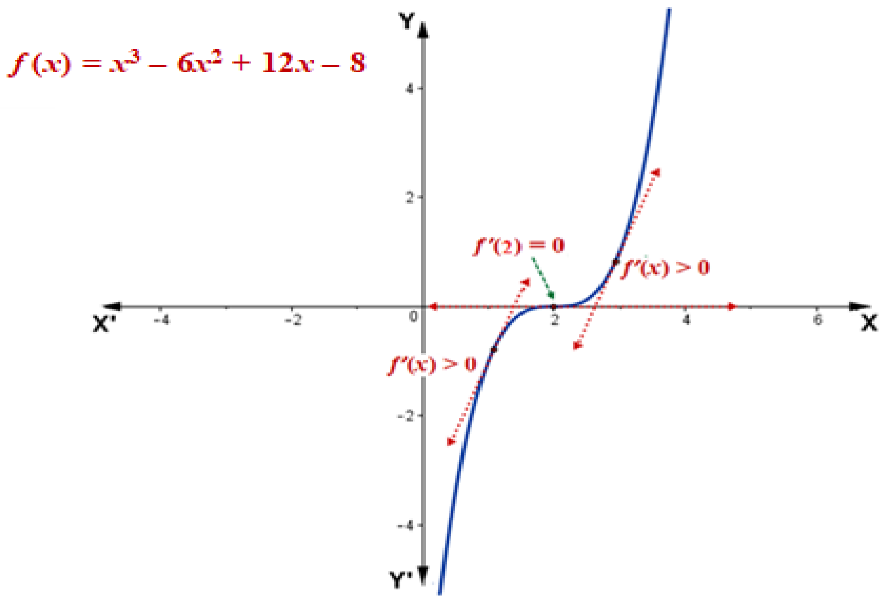

Fig. 18

We have one more test to examine local maxima and local minima of a given function. This test is comparatively easier to apply in many cases than the first derivative test.

5. Local Maxima and Local Minima (Second Derivative Test):

Let f be a function defined on an interval I and $c \in$ I.

Let, f be twice differentiable at c (i.e., second order derivative of f exists at c). Then,

(i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$.

The value $f(c)$ is local maximum value of f.

(ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$.

In this case, the value $f(c)$ is local minimum value of f.

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

In this case, we go back to the first derivative test and find whether c is a point of local maxima or local minima or a point of inflexion.

Geometrical Interpretation (Second Derivative Test) :

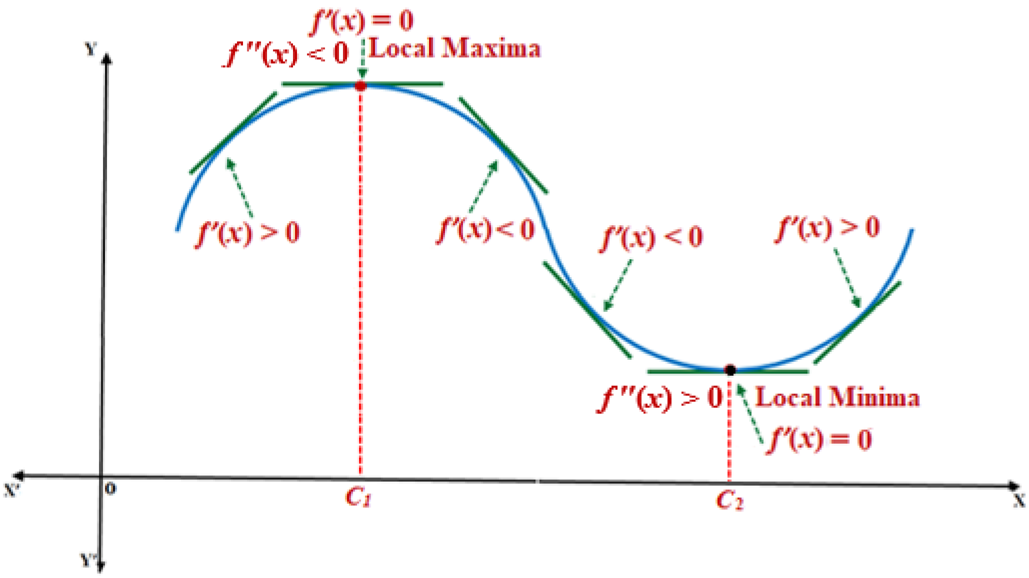

Fig. 19

In the graph of the function above, we have, local maxima at $x = c_1$ and local minima at $x = c_2$.

See on the left of $x = c_1$ we have $f'(x) > 0$, at $x = c_1$, $f'(x) = 0$ and on the right of $x = c_1$ we have $f'(x) < 0$. This shows that in the interval $(c_1 - h, c_1 + h)$, $f'(x)$ is a decreasing function, where h is a small positive number $(h > 0)$.

Since $f'(x)$ is a decreasing function at c_1, therefore its derivative at $x = c_1$, should be negative, hence we have $f''(c_1) < 0$.

Hence, for a twice differentiable function f, if at a point $x = c$ of its domain, we have $f'(c) = 0$ and $f''(c) < 0$, then $x = c$ is a point of local maxima.

Similarly from the graph shown above we observe that on the left of $x = c_2$ we have $f'(x) < 0$, at $x = c_2$, $f'(x) = 0$ and on the right of $x = c_2$ we have $f'(x) > 0$. This shows that in the interval $(c_2 - h, c_2 + h)$, $f'(x)$ is an increasing function, where h is a small positive number $(h > 0)$.

Since $f'(x)$ is an increasing function at $x = c_2$, therefore, its derivative at $x = c_2$ should be positive, so we have $f''(c_2) > 0$.

Hence, for a twice differentiable function f, if at a point $x = c$ of its domain, we have $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a point of local minima.

Example 4 :

Find all the points of local maxima and local minima of the function f given by, $f(x) = 2x^3 - 21x^2 + 36x - 20$, using second derivative test.

Solution :

The given function is,

$$f(x) = 2x^3 - 21x^2 + 36x - 20$$

Since the given function is a polynomial of third degree, so we can find both $f'(x)$ and $f''(x)$ for all $x \in \mathrm{R}$.

$$f'(x) = 6x^2 - 42x + 36$$

$$= 6(x^2 - 7x + 6) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

$$= 6(x - 1)(x - 6)$$

For local maxima and local minima and we must have,

$$f'(x) = 0$$

$$\Rightarrow \quad x = 1, 6$$

Hence, we have two critical points of the given function where we can have maxima or minima.

From (i), $\quad f''(x) = 6(2x - 7)$

At $x = 1$, $\quad f''(1) = 6(2 \text{ x } 1 - 7) = -30 < 0$

So, $x = 1$ is the point of local maxima.

Local maximum value $= f(1) = 2(1)^3 - 21(1)^2 + 36(1) - 20$

$= -3$

At $x = 6$, $f''(6) = 6(2 \times 6 - 7) = 30 > 0$

So, $x = 6$ is the point of local minima.

Local minimum value $= f(6) = 2(6)^3 - 21(6)^2 + 36(6) - 20$

$= -128$

Example 5 :

Find the points of local maxima or local minima of the function, if any, of the function given by, $f(x) = x^3 + x^2 + x + 1$.

Solution :

The given function is,

$$f(x) = x^3 + x^2 + x + 1$$

Since the given function is a polynomial, so we can find $f'(x)$.

$$f'(x) = 3x^2 + 2x + 1$$

For local maxima and local minima we must have,

$$f'(x) = 0$$

$$f'(x) = 0 \Rightarrow 3x^2 + 2x + 1 = 0$$

But the above equation gives only imaginary values of x.

Hence, we cannot get, $f'(x) = 0$, for any real value of x.

Therefore, $f(x)$ does not have any point of local maxima or local minima.

Example 6 :

Find the points of local maxima or local minima of the function, given by, $f(x) = \sin x - \cos x$, $0 < x < 2\pi$. Also find the local maximum or local minimum values as the case may be.

Solution :

The given function is,

$$f(x) = \sin x - \cos x, \quad \text{where } 0 < x < 2\pi$$

$$\Rightarrow \quad f'(x) = \cos x + \sin x \quad \ldots\ldots\ldots\ldots\ldots\ldots(i)$$

For local maxima and local minima and we must have,

$$f'(x) = 0,$$

$$\Rightarrow \cos x + \sin x = 0$$

$$\Rightarrow \sin x = -\cos x$$

$$\Rightarrow \tan x = -1 \quad \Rightarrow \quad x = \frac{3\pi}{4}, \quad \frac{7\pi}{4}$$

Thus, $x = \frac{3\pi}{4}$ and $\frac{7\pi}{4}$ are possible points of local maxima or local minima. Let us now test the function at these points.

From (*i*) $\quad f''(x) = -\sin x + \cos x$

At $x = \frac{3\pi}{4}$, $\quad f''\left(\frac{3\pi}{4}\right) = -\sin\left(\frac{3\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right)$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} < 0$$

Hence, $x = \frac{3\pi}{4}$, is the point of local maxima.

Local maximum value $= f\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) - \cos\left(\frac{3\pi}{4}\right)$

$$= \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}}$$

$$= \sqrt{2}$$

Look at the graph of the function; $f(x) = \sin x - \cos x$, below;

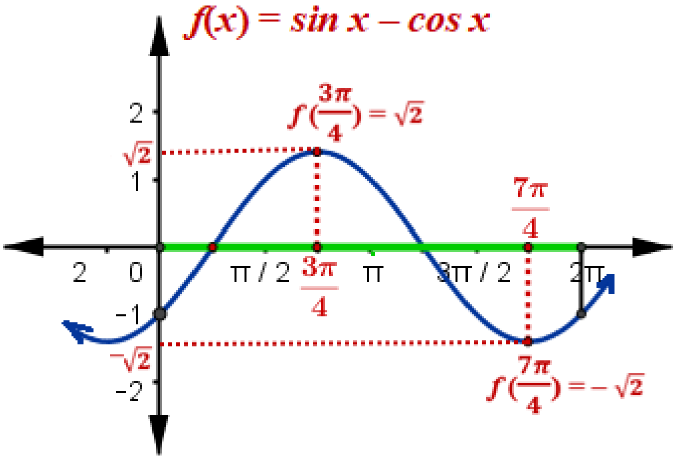

Fig. 20

At $x = \frac{7\pi}{4}$, $\quad f''\left(\frac{7\pi}{4}\right) = -\sin\left(\frac{7\pi}{4}\right) + \cos\left(\frac{7\pi}{4}\right)$

$$= -\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} > 0$$

Hence, $x = \frac{7\pi}{4}$, is the point of local minima.

Local minimum value $= f\left(\frac{7\pi}{4}\right) = sin\left(\frac{7\pi}{4}\right) - cos\left(\frac{7\pi}{4}\right)$

$$= \left(-\frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}}$$

$$= -\sqrt{2}$$

See the graph of the function, Fig. 20, locate the point of local minima and verify the local minimum value of the function.

Example 7 :

Find the critical points of the function given by,

$$f(x) = x^5 - 5x^4 + 5x^3 - 1,$$

and discuss these points for local maxima and local minima.

Solution :

The given function is,

$$f(x) = x^5 - 5x^4 + 5x^3 - 1$$

$$\Rightarrow \quad f'(x) = 5x^4 - 20x^3 + 15x^2$$

$$= 5x^2(x^2 - 4x + 3)$$

$$= 5x^2(x-1)(x-3) \quad \ldots\ldots\ldots\ldots.(i)$$

$$f'(x) = 0 \Rightarrow x = 0, 1, 3$$

Thus critical points are, $x = 0, 1, 3$

We have, $f''(x) = 20x^3 - 60x^2 + 30x$

At $x = 1$, $f''(1) = 20 - 60 + 30$

$= -10 < 0$,

Therefore, at $x = 1$, the function has local maxima.

Local maximum value at $x = 1$, $f(1) = 1 - 5 + 5 - 1 = 0$ $\ldots\ldots\ldots\ldots.(ii)$

At $x = 3$, $f''(3) = 20(3)^3 - 60(3)^2 + 30(3)$

$= 540 - 540 + 90 = 90 > 0$

Thus, at $x = 3$, the function has local minima.

Local minimum value at $x = 3$, $f(3) = (3)^5 - 5(3)^4 + 5(3)^3 - 1$

$= 243 - 405 + 135 - 1$

$= -28$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(iii)$

At $x = 0$, $f''(0) = 20(0)^3 - 60(0)^2 + 30(0) = 0$

Since, $f''(0) = 0$, the second derivative test fails. Let us now apply first derivative test and examine the point, $x = 0$, for local maxima or local minima.

From (i) we have,

$$f'(x) = 5x^2(x-1)(x-3)$$

$f'(x) > 0$, for all values of x to the left of 0 and close to 0, similarly, $f'(x) > 0$, for values of x to the right of 0 and close to 0. Therefore, by first derivative test, the point $x = 0$, is neither a point of local maxima nor a point of local minima. See the graph of the function in the Fig. 21, it explains that $x = 0$ is the point of inflexion, at $x = 1$ the function has local maxima and the function has local minima at $x = 3$.

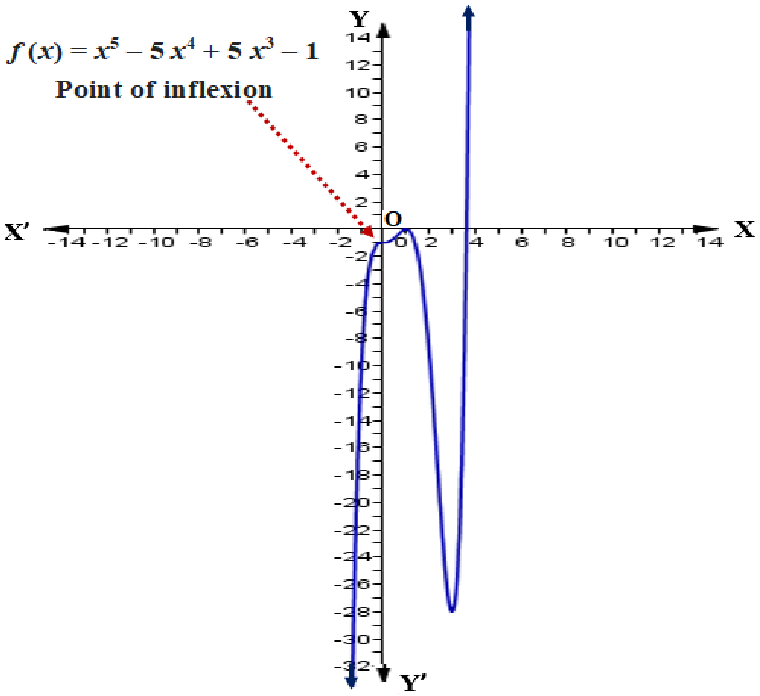

Fig. 21

Maximum and Minimum Values of a Function in a Closed Interval:

Definition:

Let f be a real valued funcion defined on an interval I, then,

(i) f is said to be absolutely maximum at a point $c \in$ I if and only if,

$f(c) \geq f(x)$, for all $x \in$ I.

(ii) f is said to be absolutely minimum at a point $c \in$ I if and only if,

$f(c) \leq f(x)$, for all $x \in$ I.

See below, in Fig. 22, the graph of a continuous function defined on a closed interval [p, q]. Observe that the function f has local minima at $x = a, c, e$ and local minimum values are $f(a)$, $f(c)$ and $f(e)$. The function has local maxima at points $x = b, d, k$ and local maximum values are $f(b)$, $f(d)$ and $f(k)$.

Also from the graph, it is evident that f has three local maximum values and three local minimum values. If we consider the closed interval $[p, q]$, then $f(d)$ is the maximum value of the function in the entire interval. Maximum value $f(d)$ of f at $x = d$ is called absolute maximum value or global maximum or greatest value of the function on the interval $[p, q]$.

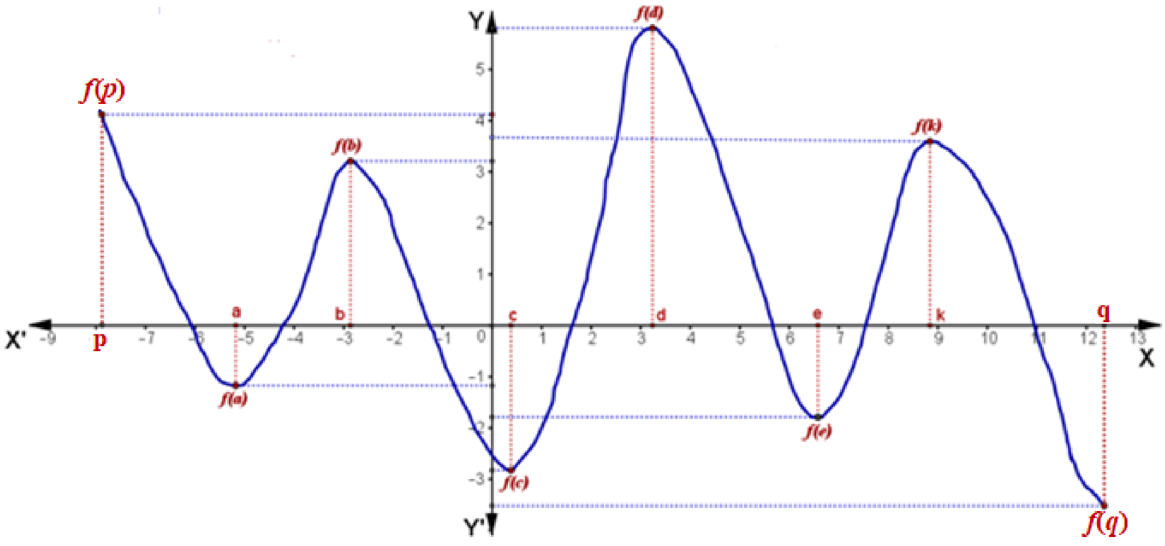

Fig. 22

We have three points of local minima but the smallest value of the function in the closed interval [p, q] is $f(q)$ at $x = q$. Minimum value $f(q)$ of f at $x = q$ is called absolute minimum value or global minimum or least value of the function on the interval [p, q].

Further note that the absolute minimum value of f is different from any of the local minimum values of f.

So, a local minimum value of a function may not be the least (absolute minimum) value of the function in a given interval. Similarly, a local maximum value of a function may not be the greatest (absolute maximum) value of the function in a given interval.

If we cosider the open interval (p, q), then absolute maximum value of f is $f(d)$ and absolute minimum value of f is $f(c)$.

Thus, if a function f is continuous on a closed interval $[a, b]$, then it attains the absolute maximum (absolute minimum) values at critical points or at the end points of the interval $[a, b]$ and in case of open interval (a, b), the function attains the absolute maximum (absolute minimum) values at the critical points of the interval (a, b).

Working Rule to find absolute maximum and absolute minimum values of a function in an interval :

Step 1: Find all critical points of f in the interval, i.e., find points x where either $f'(x) = 0$ or f is not differentiable.

Step 2: Take the end points of the interval.

Step 3: At all these points (listed in Step 1 and 2), calculate the values of f

Step 4: Identify the maximum and minimum values of f out of the values calculated in Step 3. This maximum value will be the absolute maximum (greatest) value of f and the minimum value will be the absolute minimum (least) value of f.

Example 8 :

Find the absolute maximum and absolute minimum values of a function f given by,

$f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval $[0, 3]$.

Solution :

The given function is,

$$f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25, \quad x \in [0, 3]$$

Given function is a polynomial, hence it is differentiable for all $x \in \mathrm{R}$,

$$f'(x) = 12x^3 - 24x^2 + 24x - 48, \quad x \in [0, 3]$$

$$f'(x) = 0 \Rightarrow 12x^3 - 24x^2 + 24x - 48 = 0$$

$$\Rightarrow \quad x^3 - 2x^2 + 2x - 4 = 0$$

$$\Rightarrow \quad (x - 2)(x^2 + 2) = 0$$

$$\Rightarrow \quad x = 2, \ \pm\sqrt{2}\, i$$

We got only one real value for x and other two values are imaginary.

Also, $2 \in (0, 3)$, hence we have one critical point of the function in the given interval.

To find the absolute maximum and absolute minimum values of the function, we have to compare the values of the function at critical point $x = 2$ and at the end points $x = 0$ and $x = 3$. So we need to find $f(0)$, $f(2)$ and $f(3)$.

$$f(0) = 3(0)^4 - 8(0)^3 + 12(0)^2 - 48(0) + 25 = 25$$

$$f(2) = 3(2)^4 - 8(2)^3 + 12(2)^2 - 48(2) + 25$$

$= 3 \times 16 - 8 \times 8 + 12 \times 4 - 48 \times 2 + 25 = -39$

$f(3) = 3(3)^4 - 8(3)^3 + 12(3)^2 - 48(3) + 25$

$= 3 \times 81 - 8 \times 27 + 12 \times 9 - 48 \times 3 + 25 = 16$

Comparing these values, we find that the absolute maximum value of the function is 25 and absolute minimum value of the function is – 39.

The point of absolute maximum is $x = 0$ and the point of absolute minimum is $x = 2$.

Example 9 :

Find the maximum and minimum values of function f given by,

$f(x) = x + \sin 2x$ on the interval $[0, 2\pi]$.

Solution :

The given function is,

$$f(x) = x + \sin 2x \ , \quad x \in [0, 2\pi]$$

$$f'(x) = 1 + 2\cos 2x$$

$$f'(x) = 0 \Rightarrow 1 + 2\cos 2x = 0$$

$$\Rightarrow \quad \cos 2x = \frac{-1}{2}$$

$$\Rightarrow \quad \cos 2x = \cos\frac{2\pi}{3}$$

General solution for $\cos 2x$ is,

$$2x = 2n\pi \pm \frac{2\pi}{3}$$

$$\Rightarrow \quad x = n\pi \pm \frac{\pi}{3} \qquad \text{(where } n \text{ is an integer)}$$

$$\Rightarrow \quad x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \qquad (\because x \in [0, 2\pi])$$

Thus the critical points are; $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

The graph of the function is shown in the Fig. 23, observe the critical points and locate local maxima and local minima.

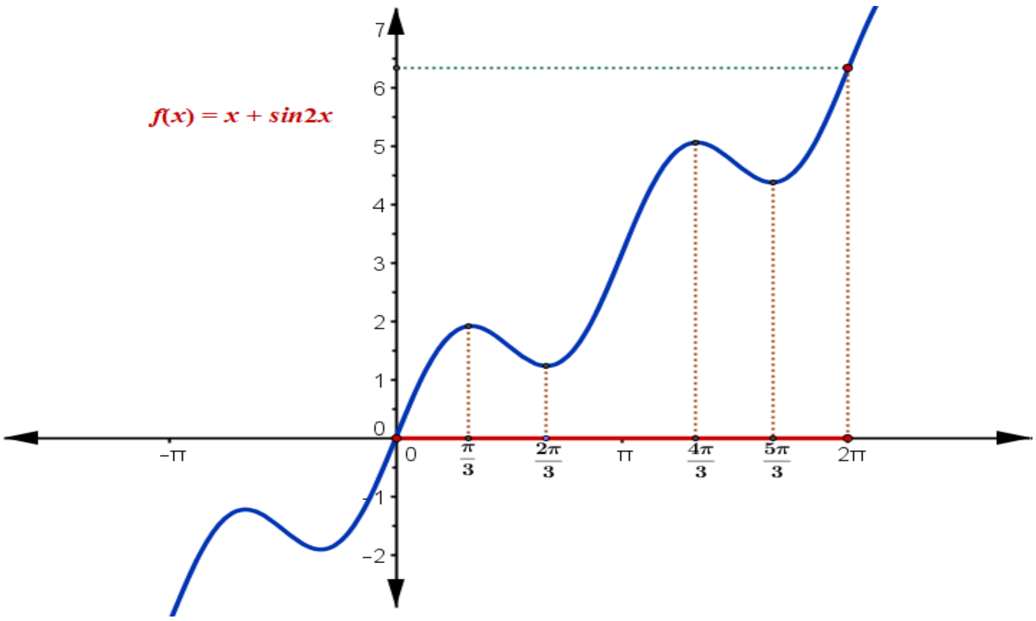

Fig. 23

To find maximum and minimum values of the function in the given interval, we have to compare the values of the function at all the critical points and also at the end points.

$f(0) = 0 + \sin 0 = 0$ ……………….(*i*)

$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$ ……………….(*ii*)

$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\frac{4\pi}{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$ ……………….(*iii*)

$f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + \sin\frac{8\pi}{3} = \frac{4\pi}{3} + \sin\left(3\pi - \frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$ ……….(*iv*)

$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sin\frac{10\pi}{3} = \frac{5\pi}{3} + \sin\left(3\pi + \frac{\pi}{3}\right) = \frac{5\pi}{3} \quad \frac{-\sqrt{3}}{2}$ ……….(*v*)

$f(2\pi) = 2\pi + \sin 4\pi = 2\pi$ ……………….(*vi*)

From all these values, the maximum value is 2 π and the minimum value is 0. The point of maximum value of the function is 2 π and the point of minimum value is 0. (See Fig. 23)

Example 10 :

Show that semi-vertical angle of right circular cone of given surface area and maximum volume is,

$\sin^{-1}\left(\frac{1}{3}\right)$.

Solution :

Let r be the radius, l the slant height and h be the height of the cone. The surface area of the cone is given, let it be S, then, S is constant and

$$S = \pi r^2 + \pi rl \quad \text{..............}(i)$$

$$\therefore \quad l = \frac{S - \pi r^2}{\pi r} \quad \text{..............}(ii)$$

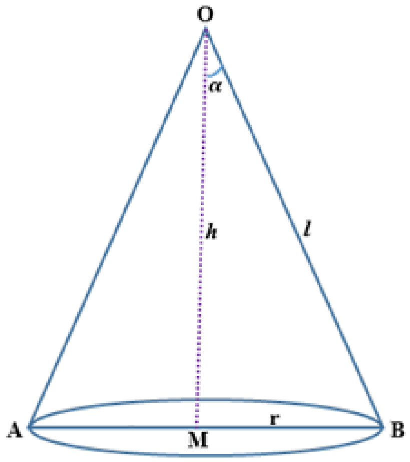

Fig. 24

Let V be the volume of the cone, then

$$V = \frac{1}{3}\pi r^2 h \quad \text{..............}(iii)$$

Form, Fig. 24, $l^2 = h^2 + r^2$ ....(iv)

From (iii) and (iv),

$$V = \frac{1}{3}\pi r^2\sqrt{l^2 - r^2} \quad \text{..............}(v)$$

We have to maximize volume V, when V is maximum, V^2 should also be maximum.

$$V^2 = \frac{1}{9}\pi^2 r^4 \left(l^2 - r^2\right)$$

$$V^2 = \frac{1}{9}\pi^2 r^4\left[\left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2\right] \quad \text{........(from } (ii))$$

$$V^2 = \frac{1}{9}\pi^2 r^4\left[\frac{\left(S - \pi r^2\right)^2 - \pi^2 r^4}{\pi^2 r^2}\right]$$

$$V^2 = \frac{1}{9}\pi^2 r^4\left[\frac{S^2 - 2S\pi r^2 + \pi^2 r^4 - \pi^2 r^4}{\pi^2 r^2}\right]$$

$$V^2 = \frac{1}{9}r^2\left(S^2 - 2S\pi r^2\right)$$

$$V^2 = \frac{1}{9}\left(S^2 r^2 - 2S\pi r^4\right) \qquad \ldots\ldots\ldots\ldots\ldots(vi)$$

Let Z = V^2, then V is maximum or minimum according as Z is maximum or minimum.

$$Z = \frac{1}{9}\left(S^2 r^2 - 2S\pi r^4\right)$$

$$\frac{dZ}{dr} = \frac{1}{9}\left(2S^2 r - 8S\pi r^3\right) \qquad \ldots\ldots\ldots\ldots\ldots(vii)$$

For maximum or minimum,

$$\frac{dZ}{dr} = 0$$

$$\Rightarrow \quad \frac{1}{9}\left(2S^2 r - 8S\pi r^3\right) = 0$$

$$\Rightarrow \quad (2Sr - 8\pi r^3) = 0$$

$$\Rightarrow \quad r\,(2S - 8\pi r^2) = 0$$

$$\Rightarrow \quad r = 0, \quad 2S - 8\pi r^2 = 0$$

But, r = 0 is not possible (*r is the radius of the cone*),

$\therefore$ $2S - 8\pi r^2 = 0$,

Hence, $\quad S = 4\pi r^2 \qquad \ldots\ldots\ldots\ldots\ldots(viii)$

Again differentiating, from equation (*vii*),

$$\frac{d^2Z}{dr^2} = \frac{1}{9}\left(2S^2 - 24S\pi r^2\right) \qquad \ldots\ldots\ldots\ldots\ldots(ix)$$

$$\left(\frac{d^2Z}{dr^2}\right)_{S=4\pi r^2} = \frac{2}{9}\left[S^2 - 12S\pi\left(\frac{S}{4\pi}\right)\right] \qquad \ldots\ldots\text{(from equation } (viii))$$

$$= \frac{2}{9}\,[S^2 - 3S^2] = \frac{-4}{9}\,S^2 < 0$$

$\therefore$ Z (*i.e.* V) is maximum, when $S = 4\pi r^2$

$$\Rightarrow \quad \pi r^2 + \pi r l = 4\pi r^2$$

$$\Rightarrow \quad \pi r l = 3\pi r^2$$

$$\Rightarrow \quad l = 3r$$

From figure, $\sin\alpha = \frac{r}{l} = \frac{1}{3}$

Hence, volume of the cone is maximum, when

$$\alpha = \sin^{-1}\left(\frac{1}{3}\right)$$

7. Applied problems in Maxima and Minima:

In our day to day life there are many situations where we deal with maximum and minimum values. Let us discuss some applied problems on maxima and minima.

Example 11:

A square piece of tin of side 18cm is to be converted into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off, so that the volume of the box is maximum possible? Also find the maximum volume of the box.

Solution:

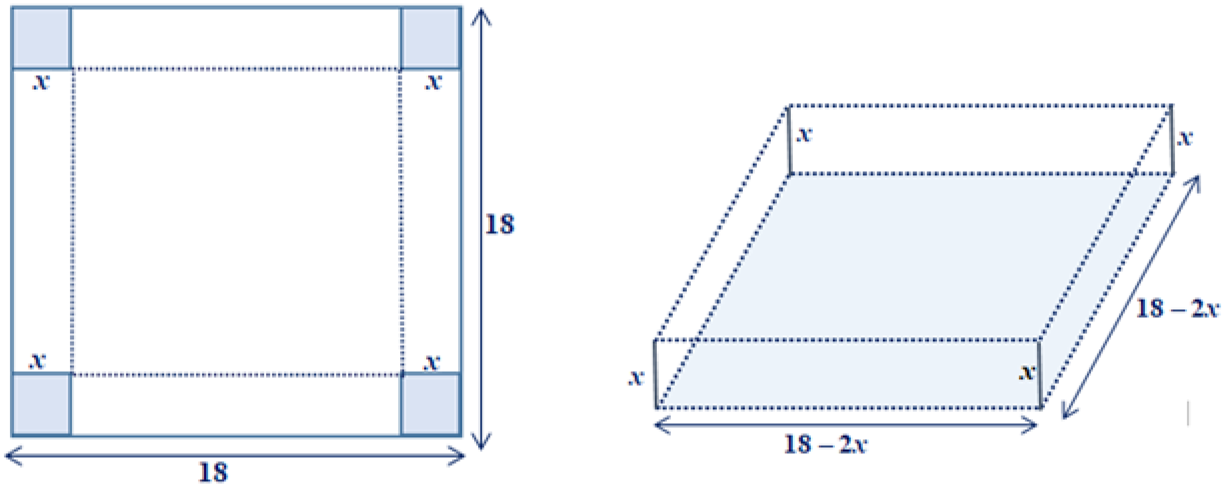

Fig. 25

Let x cm be the length of the side of removed squares. Then,

Height of the box, $h = x$,

Length of the box, $l = 18 - 2x$

Breadth of the box, $l = 18 - 2x$

Let, V be the volume of the box, then

$$V = (18 - 2x)\,(18 - 2x)\,x$$

$= 2(9 - x).\ 2(9 - x)\ x$

$= 4\ (9 - x)^2.\ x$ ………………..(i)

$= 4(x^3 - 18x^2 + 81x)$

$$\frac{dV}{dx} = 4(3x^2 - 36x + 81)$$

$= 12\ (x^2 - 12x + 27)$

For maximum volume of the box we must have,

$$\frac{dV}{dx} = 0$$

$\Rightarrow$ $12\ (x^2 - 12x + 27) = 0$

$\Rightarrow$ $12\ (x - 3)\ (x - 9) = 0$

$\Rightarrow$ $x = 3,\quad x = 9$

But, $x = 9$ is not possible, otherwise the whole tin piece will be cut in two equal parts.

Now, $\dfrac{d^2V}{dx^2} = 12\ (2x - 12) = 24\ (x - 6)$

$$\left(\frac{d^2V}{dx^2}\right)_{x=3} = 24\ (3 - 6) = -72 < 0 \quad \text{(negative)}$$

Thus, volume is maximum at $x = 3$.

Hence, a square of side 3 cm should be cut off from each corner to have the box of maximum volume.

Maximum volume of the box $= 4\ (9 - 3)^2.\ 3$ ……from (i)

$= 4.\ (6)^2.\ 3 = 432\ \text{cm}^3$

Example 12:

A helicopter of enemy is flying along the curve given by $y = x^2 + 2$. A soldier, placed at (3, 2), wants to shoot down the helicopter when it is nearest to him. Find the nearest distance.

Solution :

Let the soldier be placed at A (3, 2) and P(x, y) be the position of helicopter at any time of its flight. Since the helicopter is flying along the curve $y = x^2 + 2$, hence the helicopter's position at any time of its motion is at point $(x, x^2 + 2)$.

$\therefore$ Distance between the helicopter and the soldier

AP = $\sqrt{(x-3)^2 + (y-2)^2}$

$$= \sqrt{(x-3)^2 + ((x^2+2)-2)^2} \quad = \sqrt{(x-3)^2 + (x^2)^2}$$

$$= \sqrt{(x-3)^2 + (x)^4}$$

Let Z = $(AP)^2$, then

$$Z = (x-3)^2 + (x)^4$$

AP will be maximum or minimum according as Z is maximum or minimum. Now,

$$\frac{dZ}{dx} = 2(x-3) + 4x^3 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (i)$$

For maximum or minimum,

$$\frac{dZ}{dx} = 0 \Rightarrow 2(x-3) + 4x^3 = 0$$

$$\Rightarrow (x-3) + 2x^3 = 0$$

$$\Rightarrow 2x^3 + x - 3 = 0$$

$$\Rightarrow (x-1)(2x^2 + 2x + 3) = 0$$

$\Rightarrow \quad x = 1$, because, $2x^2 + 2x + 3 = 0$ gives imaginary values of x.

From (i), $\dfrac{d^2Z}{dx^2} = 2x + 12x^2$

$$\left(\frac{d^2Z}{dx^2}\right)_{x=1} = 2 + 12 = 14 > 0$$

Hence, Z is minimum when $x = 1$.

Putting $x = 1$ in $y = x^2 + 2$, we get, $y = 3$.

Thus, AP is minimum when helicopter is at point (1, 3) on the curve.

Nearest distance between the soldier and the helicopter

$$= \sqrt{(1-3)^2 + (3-2)^2} \quad = \sqrt{(-2)^2 + (1)^2}$$

$$= \sqrt{5}$$

Hence, the nearest distance between the soldier and the helicopter is $\sqrt{5}$

Example 13:

Manufacturer can sell x items at a price of rupees $\left(5 - \dfrac{x}{100}\right)$ each. The cost price of x items is rupees $\left(\frac{x}{5}+500\right)$. Find the number of items he should sell to earn maximum profit.

Solution :

Let S(x) be the selling price and C(x) be the cost price of x items.

Since manufacturer sells x items at a price of rupees $\left(5-\frac{x}{100}\right)$ each, therefore,

$$S(x) = \left(5-\frac{x}{100}\right).x$$

$$= \left(5x-\frac{x^2}{100}\right)$$

And the cost price of x items is rupees $\left(\frac{x}{5}+500\right)$,

Therefore, $\quad C(x) = \left(\frac{x}{5}+500\right)$

Let P(x) be the profit function, then

$$P(x) = S(x) - C(x)$$

$$= \left(5x-\frac{x^2}{100}\right)-\left(\frac{x}{5}+500\right)$$

$$= \frac{24x}{5} \quad \frac{-x^2}{100} \quad -500 \quad \ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Differentiating, $\quad \frac{d(P(x))}{dx} = \frac{24}{5} \quad \frac{-x}{50} \quad \ldots\ldots\ldots\ldots\ldots\ldots (ii)$

To earn maximum profit, $\quad \frac{d(P(x))}{dx} = 0$

$$\Rightarrow \quad \frac{24}{5} \quad \frac{-x}{50} = 0$$

$$\Rightarrow \quad \frac{x}{50} = \frac{24}{5}$$

$$\Rightarrow \quad x = 240$$

From (ii)

$$\frac{d^2(P(x))}{dx^2} = \frac{-1}{50}$$

$$\therefore \quad \left(\frac{d^2(P(x))}{dx^2}\right)_{x=240} = \frac{-1}{50} < 0$$

Thus, $x = 240$ is a point of maxima. Hence, the manufacturer can earn maximum profit, if he sells 240 items.

Example 14 :

Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

Solution :

Let r be the radius and $2x$ be the height of the cylinder ABCD which is inscribed in the given sphere of radius R. Since sphere and cylinder both are symmetrical objects hence for maximum volume of cylinder, the axis of the cylinder must be along the diameter of the sphere. Let O be the centre of the sphere, then ON = x.

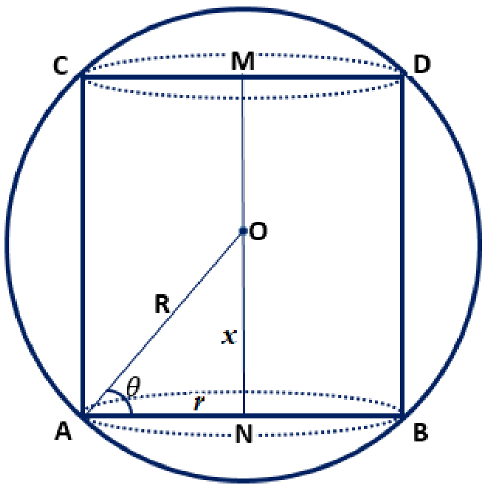

Fig. 26

We have,

$$OA^2 = ON^2 + AN^2$$

$$\Rightarrow \quad AN^2 = OA^2 - ON^2$$

$$\Rightarrow \quad AN^2 = R^2 - x^2$$

Let V be the volume of the cylinder, then

$$V = \pi\,(AN)^2\,MN$$

$$\Rightarrow \quad V = \pi\,(R^2 - x^2)\,(2x)$$

$$= 2\pi\,(R^2 x - x^3) \quad \ldots\ldots\ldots\ldots\ldots\ldots(i)$$

Differentiating,

$$\frac{dV}{dx} = 2\pi\,(R^2 - 3x^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

And $$\frac{d^2V}{dx^2} = -12\pi x \quad \ldots\ldots\ldots\ldots\ldots\ldots(iii)$$

For maximum volume of cylinder, we must have,

$$\frac{dV}{dx} = 0$$

$$\Rightarrow 2\pi\,(R^2 - 3x^2) = 0$$

$$\Rightarrow \quad x = \frac{R}{\sqrt{3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(iv)$$

From (*iii*) and (*iv*)

$$\left(\frac{d^2V}{dx^2}\right)_{x=\frac{R}{\sqrt{3}}} = -12\pi\left(\frac{R}{\sqrt{3}}\right) < 0$$

Hence, volume of the cylinder is maximum when $x = \dfrac{R}{\sqrt{3}}$.

Height of the cylinder for maximum volume

$$MN = 2x = \frac{2R}{\sqrt{3}}\ .$$

Example 15:

A window is in the form of a rectangle surmounted by a semicircular opening. If total perimeter of the window is P, find the dimensions of the window to admit maximum light through the whole opening.

Solution :

Let ABCD be the rectangle of the window surmounted by the semicircle on side AB as diameter.

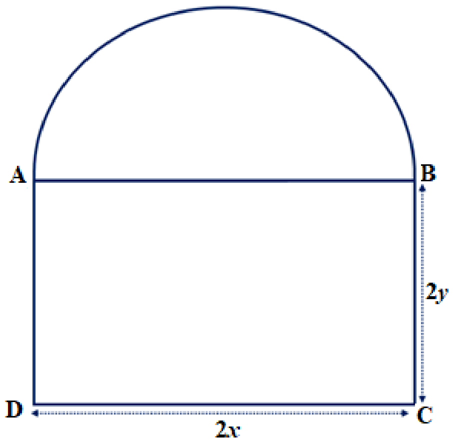

Fig. 27

Let, length of the rectangle, AB = $2x$

and breadth of the rectangle, AD = $2y$

Total perimeter of the window = P

$\therefore$ $P = 2y + 2x + 2y + \pi x$ ($\because$ radius of the semicircle is x)

$= 2x + 4y + \pi x$ ……………………(i)

Let A be the area of whole opening of the window, then

$$A = (2x)(2y) + \frac{\pi x^2}{2}$$

$$= (4xy) + \frac{\pi x^2}{2} \qquad \text{……………………}(ii)$$

Eliminating y using (i) we get,

$$A = x(P - 2x - \pi x) + \frac{\pi x^2}{2}$$

$$= Px - 2x^2 - \pi x^2 + \frac{\pi x^2}{2}$$

$$= Px - 2x^2 - \frac{\pi x^2}{2}$$

Differentiating,

$$\frac{dA}{dx} = P - 2.2x - \frac{2\pi x}{2}$$

$$= P - 4x - \pi x \qquad \text{……………………}(iii)$$

To admit maximum or minimum light through the window we must have,

$$\frac{dA}{dx} = 0 \Rightarrow P - 4x - \pi x = 0$$

$$\Rightarrow P - x(4 + \pi) = 0$$

$$\Rightarrow x = \frac{P}{(4+\pi)}$$

From (*iii*),

$$\left(\frac{d^2A}{dx^2}\right) = -4 - \pi < 0$$

Thus, window will admit maximum light through the whole opening

When, $x = \frac{P}{(4+\pi)}$

Putting value of x in (*i*) we get,

$$4y = P - (2 + \pi)\left(\frac{P}{(4+\pi)}\right)$$

$$= \frac{(4+\pi)P - (2+\pi)P}{(4+\pi)P}$$

$$= \frac{2P}{(4+\pi)}$$

$$\Rightarrow y = \frac{P}{2(4+\pi)}$$

Thus,

Length of the rectangle = $2x = \frac{2P}{(4+\pi)}$

Breadth of the rectangle = $2y = \frac{P}{(4+\pi)}$

8. Summary :

1. Let f be a function defined on an interval I. Then
   - f is said to have a maximum value in I, if there exists a point c in I such that $f(c) \geq f(x)$, for all $x \in$ I.

   The number $f(c)$ is called the maximum value of f in I and the point c is called a point of maximum value of f in I.

- f is said to have a minimum value in I, if there exists a point c in I such that $f(c) \leq f(x)$, for all $x \in$ I.

The number $f(c)$, is called the minimum value of f in I and the point c, is called a point of minimum value of f in I.

- f is said to have an extreme value in I if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I. The number $f(c)$ is called an extreme value of f in I and the point c is called an extreme point.

2. Let, f be a real valued function and c be an interior point in the domain of f. Then,

(a) c is called a point of local maxima, if there exists a real number h, $h > 0$, such that

$f(c) > f(x)$, for all $x \in (c - h, c + h)$

The value $f(c)$ is called the local maximum value of f.

(b) c is called a point of local minima, if there exists a real number h, $h > 0$, such that

$f(c) < f(x)$, for all $x \in (c - h, c + h)$

The value $f(c)$ is called the local minimum value of f.

3. **First Derivative Test** for Local Maxima and Local Minima;

Let f be a function defined on an open interval I and let f be continuous at a critical point c in I. Then,

(a) If $f'(x)$ changes sign from positive to negative as x increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to c and to the left of c, and $f'(x) < 0$ at every point sufficiently close to c and to the right of c, then c is a point of local maxima.

(b) If $f'(x)$ changes sign from negative to positive as x increases through c, i.e., if $f'(x) < 0$ at every point sufficiently close to c and to the left of c, and $f'(x) > 0$ at every point sufficiently close to c and to the right of c, then c is a point of local minima.

(c) If $f'(c) = 0$ and $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima and such a point is called point of inflection.

4. **Second Derivative Test** for Local Maxima and Local Minima

Let f be a function defined on an interval I and $c \in$ I. Let, f be twice differentiable at c (i.e., second order derivative of f exists at c) then,

(a) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$.

The value $f(c)$ is local maximum value of f.

(b) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$.

In this case, the value $f(c)$ is local minimum value of f.

(c) The test fails if $f'(c) = 0$ and $f''(c) = 0$. In this case, we go back to the first derivative test and find whether c is a point of local maxima or local minima or a point of inflexion.

5. Maximum and Minimum Values of a Function in a Closed Interval:

Let f be a real valued funcion defined on a closed interval I, then,

(i) f is said to be absolutely maximum at a point $c \in$ I if and only if,

$f(c) \geq f(x)$, for all $x \in$ I.

(ii) f is said to be absolutely minimum at a point $c \in$ I if and only if,

$f(c) \leq f(x)$, for all $x \in$ I.

6. Working Rule to find absolute maximum and absolute minimum:

Step 1: Find all critical points of f in the interval, i.e., find points x where either $f'(x) = 0$ or f is not differentiable.

Step 2: Take the end points of the interval.

Step 3: At all these points (listed in Step 1 and 2), calculate the values of f

Step 4: Identify the maximum and minimum values of f out of the values calculated in Step 3. The maximum value will be the absolute maximum (greatest) value of f and the minimum value will be the absolute minimum (least) value of f.