1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 03 (Class XII, Semester - 1)
Module Name/Title	Application of Derivatives – Part 2
Module Id	lemh_10602
Pre-requisites	Rate of change of quantities, Knowledge of plotting functions
Objectives	After going through this lesson, the learners will be able to understand the following: • Assign the function as increasing, decreasing or with no change by finding derivative • Finding conditions and intervals under which a function is increasing or decreasing • Some definitions and theorems
Keywords	Monotonicity, increasing and decreasing function, domain of the function

2. Development Team

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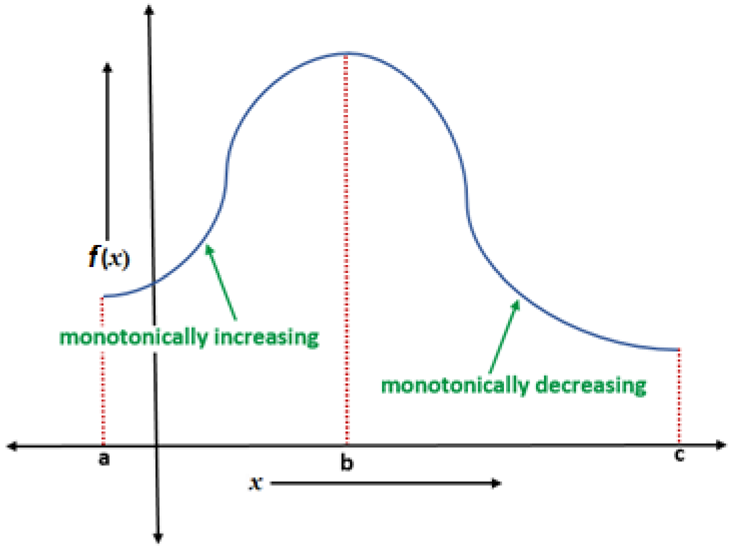
1. Introduction

In this module we are going to learn application of derivatives to find monotonicity of functions. Monotonicity means with no change. A function $f(x)$ is said to be a monotonically increasing function on [a, b] if values of $f(x)$ increase with the increase in the value of independent variable x ; $\in x$ [a, b] and monotonically decreasing function on [a, b] if values of $f(x)$ decrease with the increase in the value of x ; $x \in$ [a, b].

Observe as we move from left to right, in the figure below, the value of the independent variable x increases, $x \in$ [a, b] and the height of the graph also increases. This means the value of the function increases when x increases for $x \in$ [a, b], hence, the function is monotonically increasing on [a, b]. For $x \in$ [b, c] when we move from left to right *i.e.* when the value of the independent variable x increases the height of the graph decreases *i.e.* the value of the function decreases, hence, the function is monotonically decreasing on [b, c].



The monotonicity of any function $f(x)$ in any interval I is strongly connected to the sign of its derivative $f'(x)$ in I.

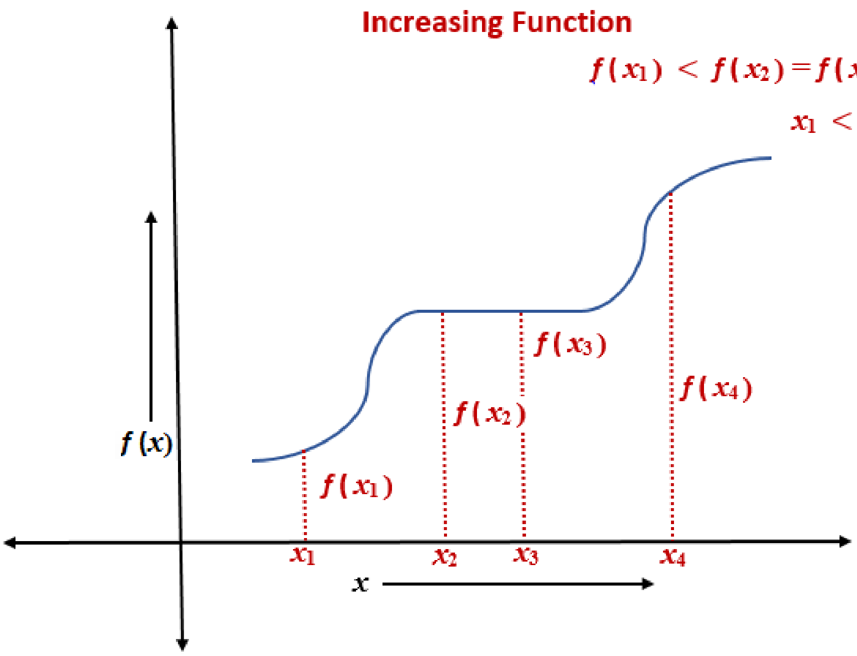
In this module we will be discussing the relation between the monotonicity of the function $f(x)$ and the sign of its derivative $f'(x)$. To determine the intervals of the monotonicity of a function in its domain we will be required to solve the inequalities,

$$f'(x) > 0, \qquad f'(x) \geq 0, \qquad f'(x) < 0, \qquad f'(x) \leq 0.$$

2. Increasing and decreasing functions

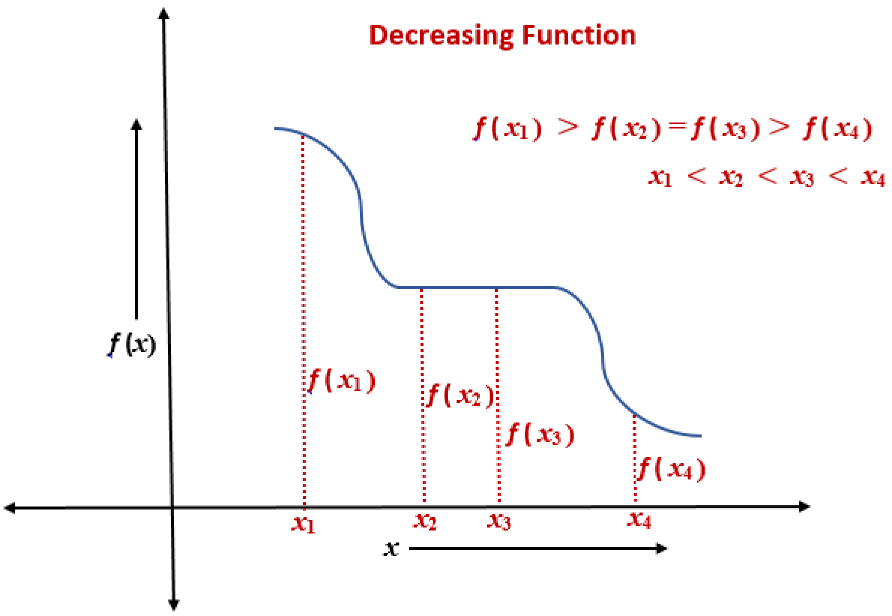
Let $f(x)$ be a real valued function defined on an interval I, then $f(x)$ is said to be an increasing function on I, if and only if,

$$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \quad \forall\, x_1, x_2 \in \mathrm{I}$$



Similarly a real valued function $f(x)$, defined on interval I is said to be a decreasing function on I, if and only if,

$$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2) \quad \forall\, x_1, x_2 \in \mathrm{I}$$



3. Some definitions

Definition-I

In the above graphs we see that at some places the graph of the function is parallel to x-axis, *i.e.* the value of the function remains constant. Let us further clarify the definitions of increasing and decreasing functions. Let I be an open interval contained in the domain of a real valued function $f(x)$. Then $f(x)$ is said to be,

(i) Increasing on I if ,

$x_1 < x_2$ in I $\Rightarrow f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in I$

(ii) Strictly increasing on I if ,

$x_1 < x_2$ in I $\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$

(iii) Decreasing on I if ,

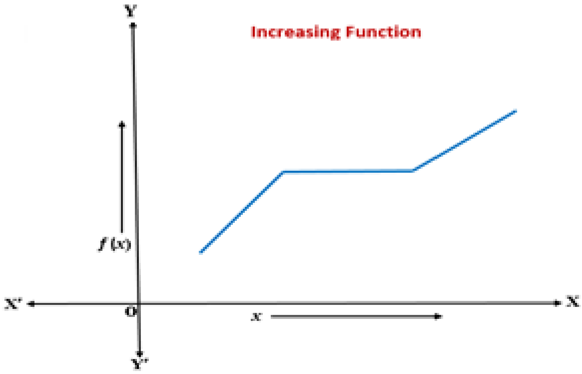
$x_1 < x_2$ in I $\Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in I$

(iv) Strictly decreasing on I if ,

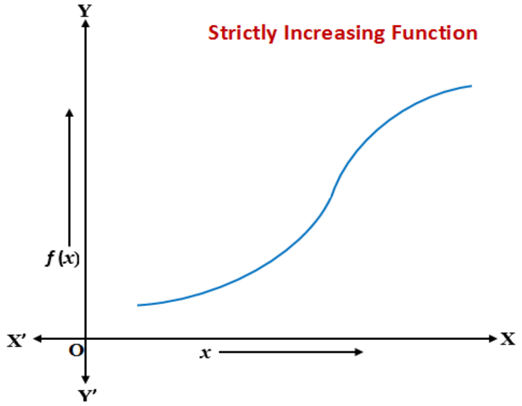
$x_1 < x_2$ in I $\Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$

See the following graphs to understand the definitions;

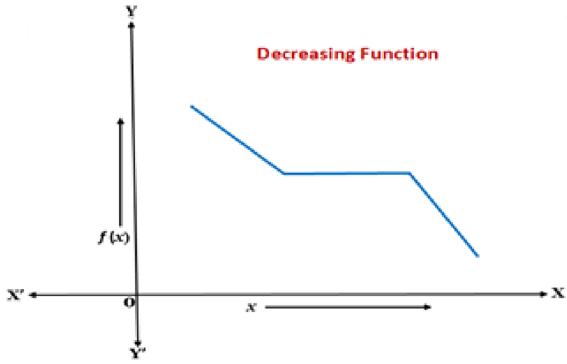
(i)


Y
Increasing Function
f (x)
X'
O
x
X
Y'

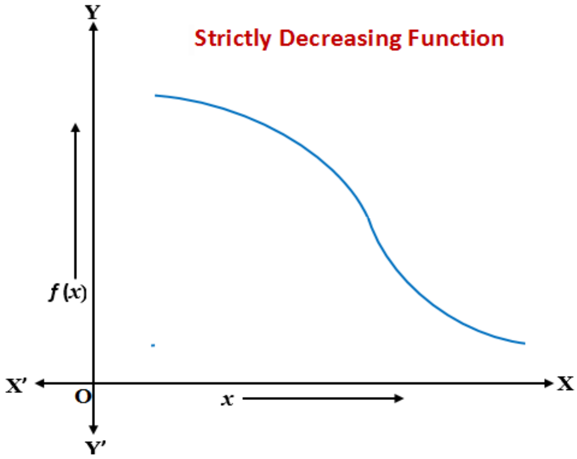
(ii)

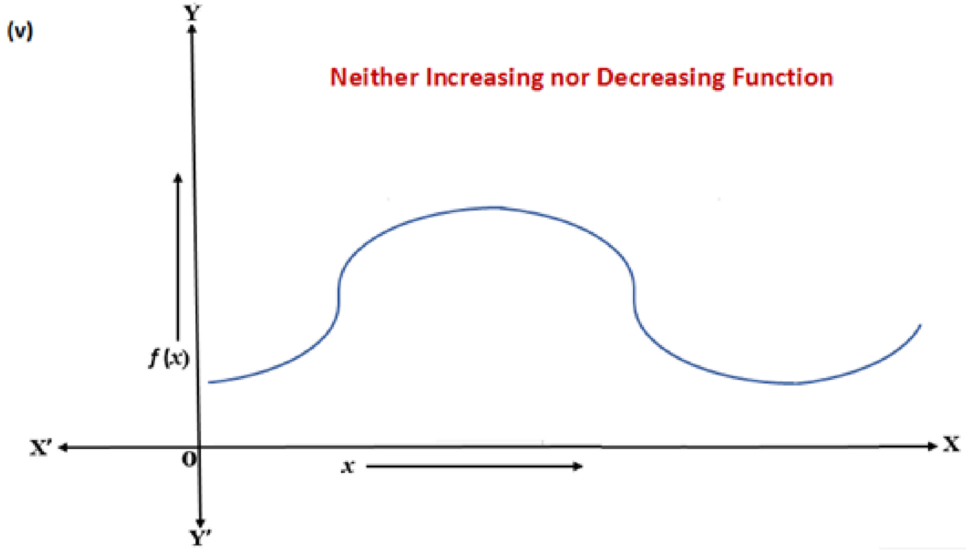

Y
Strictly Increasing Function
f (x)
X'
O
x
X
Y'

(iii)


Y
Decreasing Function
f (x)
X'
O
x
X
Y'

(iv)


Y
Strictly Decreasing Function
f (x)
X'
O
x
X
Y'



In the graph of the function $f(x)$ above, the function is increasing as well as decreasing in the interval shown in the figure. In such cases, we say that the function is neither increasing nor decreasing on the interval.

Let us now define when a function is increasing or decreasing at a point.

Definition-II

Let x_0 be a point in the domain of a real valued function $f(x)$. Then $f(x)$ is said to be increasing, strictly increasing, decreasing or strictly decreasing at x_0 if there exists an open interval I containing x_0 such that $f(x)$ is increasing, strictly increasing, decreasing or strictly decreasing, respectively, in I.

Let us clarify this definition for the case of increasing function.

A function $f(x)$ is said to be increasing at x_0 if there exists an interval

$I = (x_0 - h, x_0 + h)$, $h > 0$ such that for $x_1, x_2 \in I$,

$$x_1 < x_2 \text{ in I} \Rightarrow f(x_1) \leq f(x_2)$$

Similar statements can be given for other cases also.

Example 1: Show that the function $f(x) = 5x + 9$ is strictly increasing on R.

Solution:

Let x_1 and x_2 be any two numbers in R.

Then,

$x_1 < x_2$ in R $\Rightarrow$ $5x_1 < 5x_2$

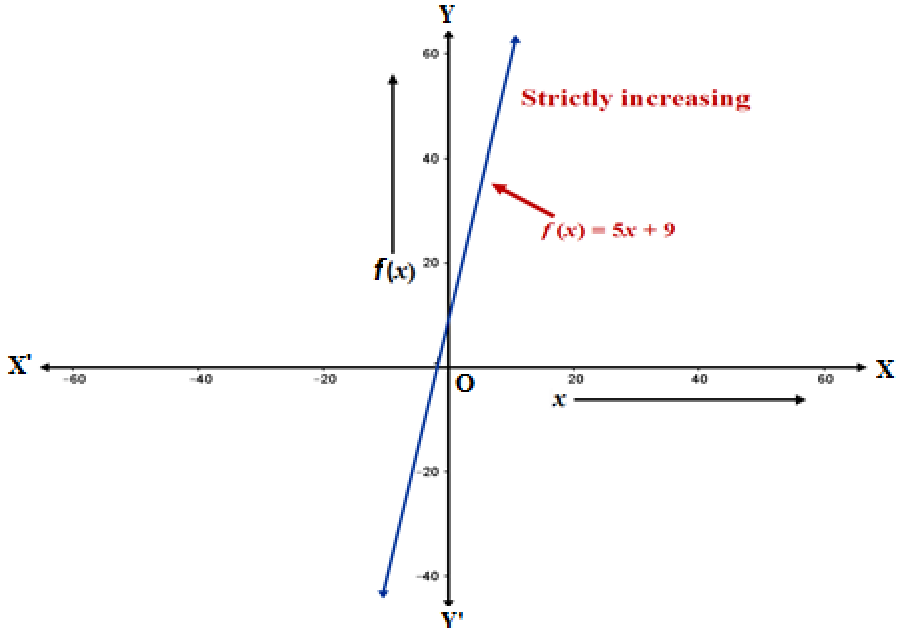
$\Rightarrow$ $5x_1 + 9 < 5x_2 + 9$

$\Rightarrow$ $f(x_1) < f(x_2)$

(for all $x_1, x_2 \in$ R)

Thus, by Definition-I, we can say that $f(x)$ is strictly increasing on R.

See the figure below, the graph of the function $f(x) = 5x + 9$ shows that the function is a strictly increasing function.



Example 2: Show that the function $f(x) = -2x + 3$ is strictly decreasing on R.

Solution:

Let x_1 and x_2 be any two numbers in R.

Then,

$x_1 < x_2$ in R $\Rightarrow$ $-2x_1 > -2x_2$

$\Rightarrow$ $-2x_1 + 3 > -2x_2 + 3$

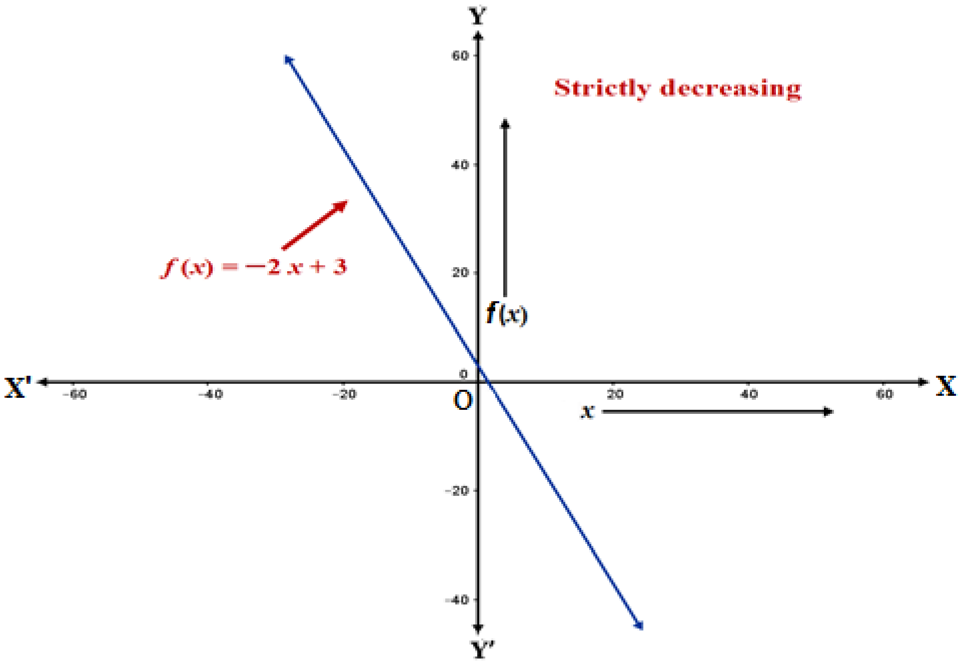
$\Rightarrow$ $f(x_1) > f(x_2)$

Thus, $x_1 < x_2 \in$ R $\Rightarrow$ $f(x_1) > f(x_2)$

(for all $x_1, x_2 \in$ R)

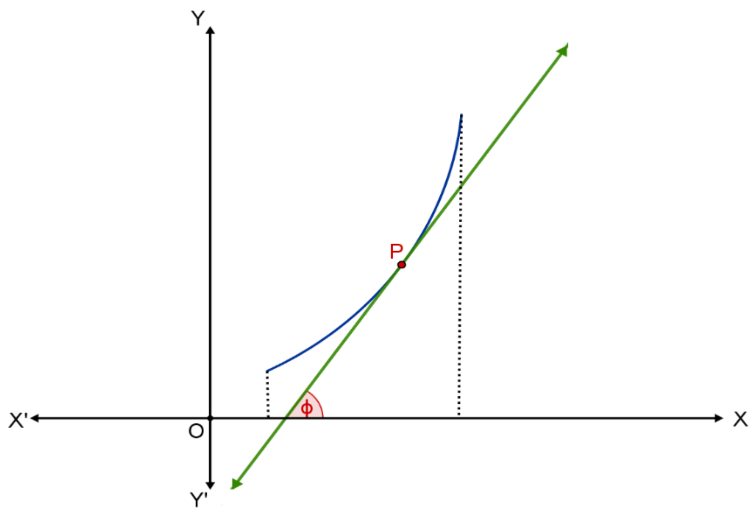
Thus, by Definition-I, we can say that $f(x)$ is a strictly decreasing function on R.

See the figure below, the graph of the function $f(x) = -2x + 3$ shows that the function is a strictly decreasing function.



Conditions for an increasing or decreasing function:

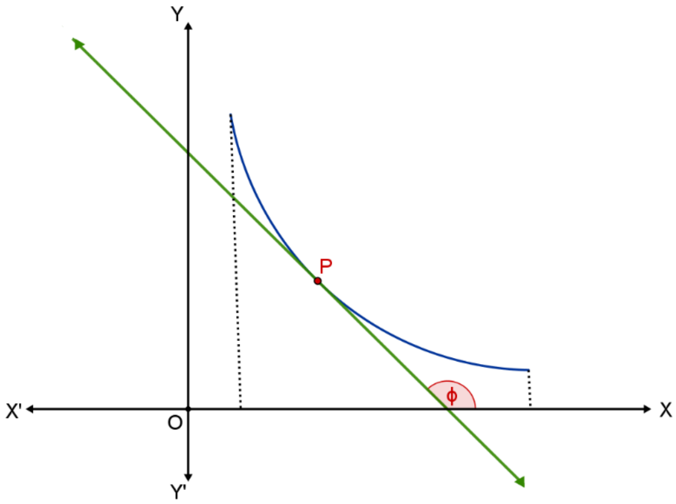
Let us now learn how to use derivatives to know where a function is increasing and where it is decreasing. We know that if derivative of a function at any point P on its curve exists then it represents the slope of the tangent to the curve at that point.



In the figure above we have the graph of a function which is strictly increasing in the interval shown, P is a point on the curve of the function. Tangent to the curve at point P is making an acute angle φ with the positive direction of x-axis. If we move point P along the curve we will find that for each position of point P in the given interval, the tangent to the curve at P will always make an acute angle φ with the positive direction of x-axis. Hence tan φ > 0 for each position of point P on the curve in the given interval.

We know that $f'(x) = \tan \varphi$, therefore, $f'(x) > 0$ for each position of point P in the given interval.

Let us now consider the graph of a function which is strictly decreasing in a given interval.



The tangent to the curve at every point P of the given interval will make an obtuse angle φ with the positive direction of x-axis. Hence tan φ > 0 for each position of point P on the curve of a function which is strictly decreasing.

Since, $f'(x) = \tan \varphi$, therefore, $f'(x) < 0$ for each position of point P on the curve in the interval.

Common sense tells that a function is increasing if its rate of change (derivative) is positive and decreasing when its rate of change is negative.

<u>Theorem:</u>

Let $f(x)$ be a function which is continuous on closed interval [a, b] and differentiable on the open interval (a, b). Then,

(a) $f(x)$ is strictly increasing in [a, b] if,

$$f'(x) > 0 \text{ for each } x \in (a, b)$$

(b) $f(x)$ is strictly decreasing in [a, b] if,

$f'(x) < 0$ for each $x \in (a, b)$

(c) $f(x)$ is a constant function in [a, b] if,

$f'(x) = 0$ for each $x \in (a, b)$

Proof :

(a)

Let $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$.

Then, by Mean Value Theorem, there exists a point c between x_1 and x_2 such that,ş

$$f(x_2) - f(x_1) = f'(c) \cdot (x_2 - x_1)$$

since, $f'(c) > 0$ as $c \in (a, b)$

(because, $f'(x) > 0$ for each $x \in (a, b)$)

therefore, $f(x_2) - f(x_1) > 0$ ($\because x_1 < x_2$)

Hence, $f(x_2) > f(x_1)$

Thus, we have

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \quad \text{for all } x_1, x_2 \in [a, b]$$

Hence, $f(x)$ is an increasing function in [a, b]. Similarly, we can prove parts (b) and (c) also.

Conclusion:

If a function $f(x)$ is continuous on [a, b] and differentiable on (a, b), then

I. $f(x)$ is strictly increasing on (a, b) if $f'(x) > 0$ for each $x \in (a, b)$

II. $f(x)$ is increasing on (a, b) if $f'(x) \geq 0$ for each $x \in (a, b)$

III. $f(x)$ is strictly decreasing on (a, b) if $f'(x) < 0$ for each $x \in (a, b)$

IV. $f(x)$ is decreasing on (a, b) if $f'(x) \leq 0$ for each $x \in (a, b)$

V. $f(x)$ is a constant function on (a, b) if $f'(x) = 0$ for each $x \in (a, b)$

VI. A function $f(x)$ will be increasing (decreasing) in R if it is so in every interval of R.

Example 3: Show that the function given by $f(x) = x^3 - 6x^2 + 7x$, $x \in R$ is strictly increasing on R.

Solution:

We have,

$$f(x) = x^3 - 6x^2 + 17x + 2$$

Differentiating with respect to x, we get

$$f'(x) = 3x^2 - 12x + 17$$

$$= 3(x^2 - 4x) + 17$$

$$= 3(x^2 - 4x + 4) + 5$$

$$= 3(x - 2)^2 + 5 > 0, \quad \text{for all } x \in R$$

Hence, $f'(x) > 0$ for all $x \in R$

Therefore, the function $f(x)$ is strictly increasing on R.

Example 4: Show that the function given by $f(x) = \cos^2 x$, is strictly decreasing on $(0, \frac{\pi}{2})$.

Solution:

We have, $f(x) = \cos^2 x \qquad x \in (0, \frac{\pi}{2})$

$\Rightarrow \qquad f'(x) = -2\cos x . \sin x$

$\qquad = -\sin 2x, \qquad x \in (0, \frac{\pi}{2})$

$x \in (0, \frac{\pi}{2}) \Rightarrow 2x \in (0, \pi)$

$\Rightarrow \sin 2x > 0$

$(\because 0 < \sin\theta < 1, \quad \text{when } \theta \in (0, \pi))$

$\Rightarrow \qquad -\sin 2x < 0$

$\Rightarrow \qquad f'(x) < 0 \quad \text{for all } x \in (0, \frac{\pi}{2})$

Hence, $f(x) = \cos^2 x$ is strictly decreasing on $(0, \frac{\pi}{2})$.

4. Finding intervals in which a function is increasing or decreasing

To find intervals in which a given function is increasing or decreasing,

i) put function equal to $f(x)$

ii) find $f'(x)$

iii) solution of inequation obtained from $f'(x) > 0$ will yield the intervals in which the function is increasing and solution of inequation obtained from $f'(x) < 0$ will give the intervals in which the function is decreasing.

Let us understand it with the help of certain examples.

Example 5: Find the intervals in which the function $f(x)$ given by,

$$f(x) = 2x^3 - 9x^2 + 12x + 15 \quad \text{is}$$

(a) strictly increasing

(b) strictly decreasing

Solution:

We have,

$$f(x) = 2x^3 - 9x^2 + 12x + 15$$

differentiating, $f'(x) = 6x^2 - 18x + 12$

$$= 6(x^2 - 3x + 2)$$

$$= 6(x-1)(x-2) \quad \ldots\ldots\ldots\ldots \text{(i)}$$

(a) for $f(x)$ to be an strictly increasing function we should have,

$$f'(x) > 0$$

$\Rightarrow$ $6(x-1)(x-2) > 0$ from (i)

$\Rightarrow$ $(x-1)(x-2) > 0$

$\Rightarrow$ $x < 1$ or $x > 2$

$\Rightarrow$ $x \in (-\infty, 1) \cup (2, \infty)$

Hence, the given function $f(x)$ is strictly increasing on $(-\infty, 1) \cup (2, \infty)$

(b) for $f(x)$ to be a strictly decreasing function we have,

$$f'(x) < 0$$

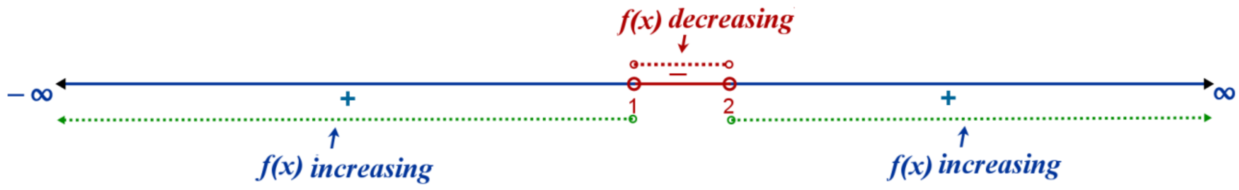
$\Rightarrow$ $6(x-1)(x-2) < 0$ from (i)

$\Rightarrow$ $(x-1)(x-2) < 0$

$\Rightarrow$ $1 < x < 2$

$\Rightarrow$ $x \in (1, 2)$

Hence, the given function $f(x)$ is strictly decreasing on $(1, 2)$.



Example 6: Find the intervals in which the function $f(x)$ given by, $f(x) = \cos 3x, \quad x \in [0, \frac{\pi}{2}]$ is (a) increasing (b) decreasing.

Solution:

We have

$$f(x) = \cos 3x$$

$$f'(x) = -3 \cos 3x$$

To find the intervals in which the function $f(x)$ is increasing or decreasing we put,

$$f'(x) = 0$$

$$\Rightarrow \quad -3 \sin 3x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$x \in [0, \tfrac{\pi}{2}] \;\Rightarrow\; 3x \in [0, \tfrac{3\pi}{2}]$$

from (i) $\quad \sin 3x = 0$

$$\Rightarrow \quad 3x = 0, \pi$$

$$\Rightarrow \quad x = 0, \frac{\pi}{3}$$

Hence, we get two disjoint intervals $[0, \frac{\pi}{3})$ and $(\frac{\pi}{3}, \frac{\pi}{2}]$

(i) $\quad x \in (0, \frac{\pi}{3}) \;\Rightarrow\; 0 < x < \frac{\pi}{3} \;\Rightarrow\; 0 < 3x < \pi$

$\Rightarrow \quad \sin 3x$ is positive when $x \in (0, \frac{\pi}{3})$

hence, $\quad f'(x) = -3 \sin 3x$ is negative when $x \in (0, \frac{\pi}{3})$

$\Rightarrow \quad f'(x) < 0$ when $x \in (0, \frac{\pi}{3})$

Hence, function $f(x) = \cos 3x$ is decreasing when $x \in (0, \frac{\pi}{3})$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (ii)

(ii) $\quad x \in (\frac{\pi}{3}, \frac{\pi}{2}) \;\Rightarrow\; \frac{\pi}{3} < x < \frac{\pi}{2} \;\Rightarrow\; \pi < 3x < \frac{3\pi}{2}$

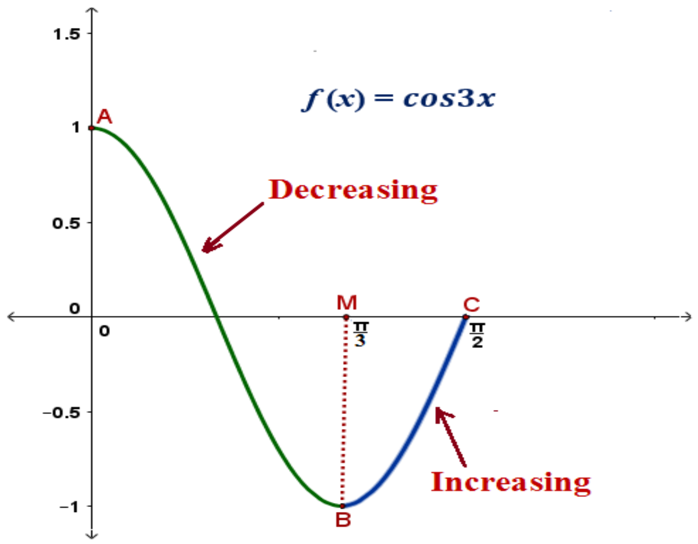
$\Rightarrow \quad \sin 3x$ is negative when $x \in (\frac{\pi}{3}, \frac{\pi}{2})$

hence, $f'(x) = -3 \sin 3x$ is positive when $x \in (\frac{\pi}{3}, \frac{\pi}{2})$

$\Rightarrow$ $f'(x) > 0$ when $x \in (\frac{\pi}{3}, \frac{\pi}{2})$

Hence, function $f(x) = \cos 3x$ is increasing when $x \in (\frac{\pi}{3}, \frac{\pi}{2})$

…………………. (iii)



We know that $f(x) = \cos 3x$ is a continuous function on $[0, \frac{\pi}{2}]$ hence combining it with results (ii) and (iii) we conclude that function $f(x) = \cos 3x$ is decreasing in $[0, \frac{\pi}{3}]$ and increasing in $[\frac{\pi}{3}, \frac{\pi}{2}]$.

Example 7: Find the intervals in which the function, $f(x) = \sin x - \cos x$, $0 \le x \le 2\pi$ is strictly increasing or strictly decreasing.

Solution

We have

$$f(x) = \sin x - \cos x, \qquad 0 \le x \le 2\pi$$

or $$f'(x) = \cos x + \sin x$$

$\Rightarrow$ $$f'(x) = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)$$

$$= \sqrt{2}[\sin\frac{\pi}{4}\cos x + \cos\frac{\pi}{4}\sin x)\ ]$$

$= \sqrt{2}\ [\ sin\left(\frac{\pi}{4} + x\right)]$

$= \sqrt{2}\ [\ sin\left(x + \frac{\pi}{4}\right)]$

Given that, $0 \le x \le 2\pi$

$\Rightarrow \quad \frac{\pi}{4} \le x + \frac{\pi}{4} \le 2\pi + \frac{\pi}{4}$ (adding $\frac{\pi}{4}$ on both sides)

$\Rightarrow \quad \frac{\pi}{4} \le x + \frac{\pi}{4} \le \frac{9\pi}{4}$ ………………… (i)

(a) for $f(x)$ to be an strictly increasing function we should have,

$f'(x) > 0$

$f'(x) > 0 \Rightarrow sin\left(x + \frac{\pi}{4}\right) > 0$

Since Sine function is positive in 1st and 2nd quadrant,

Therefore, $\frac{\pi}{4} < x + \frac{\pi}{4} < \pi$

$\Rightarrow \quad 0 < x < \frac{3\pi}{4}$ ………………… (ii)

(b) for $f(x)$ to be a strictly decreasing function we have,

$f'(x) < 0$

$\Rightarrow sin\left(x + \frac{\pi}{4}\right) < 0$

$\Rightarrow \quad \pi < x + \frac{\pi}{4} < 2\pi$

$\Rightarrow \quad \frac{3\pi}{4} < x < \frac{7\pi}{4}$ ………………… (iii)

(c) Now, the remaining interval

$2\pi \le x + \frac{\pi}{4} \le 2\pi + \frac{\pi}{4} \Rightarrow \frac{7\pi}{4} < x\ 2\pi$

$x + \frac{\pi}{4}$ lies in first quadrant,

$\Rightarrow sin\left(x + \frac{\pi}{4}\right) > 0 \Rightarrow f'(x) > 0$

Thus, $f(x)$ is strictly increasing function when,

$\frac{7\pi}{4} < x < 2\pi$ ………………… (iv)

Combining (ii), (iii) and (iv) we get,

$f(x)$ is strictly increasing in $[0, \frac{3\pi}{4}] \cup \left[\frac{7\pi}{4}, 2\pi\right]$

and $f(x)$ is strictly decreasing in $[\frac{3\pi}{4}, \frac{7\pi}{4}]$.

5. Summary

1) A real valued function $f(x)$, defined on an interval I, is said to be an increasing function on I, if and only if,

$$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \quad \forall\ x_1,\ x_2 \in I$$

2) And function $f(x)$ is strictly increasing function on I, if,

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \quad \forall\ x_1,\ x_2 \in I$$

3) A real valued function $f(x)$, defined on an interval I, is said to be a decreasing function on I, if and only if,

$$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2) \quad \forall\ x_1,\ x_2 \in I$$

4) And function $f(x)$ is strictly decreasing function on I, if,

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \quad \forall\ x_1,\ x_2 \in I$$

5) We use derivative of a function to find out the intervals in the domain of the function where the given function is increasing or decreasing.

6) If a function $f(x)$ is continuous on [a, b] and differentiable on (a, b), then

   (i) $f(x)$ is strictly increasing on (a, b) if $f'(x) > 0$ for each $x \in (a, b)$

   (ii) $f(x)$ is increasing on (a, b) if $f'(x) \geq 0$ for each $x \in (a, b)$

   (iii) $f(x)$ is strictly decreasing on (a, b) if $f'(x) < 0$ for each $x \in (a, b)$

   (iv) $f(x)$ is decreasing on (a, b) if $f'(x) \leq 0$ for each $x \in (a, b)$

   (v) $f(x)$ is a constant function on (a, b) if $f'(x) = 0$ for each $x \in (a, b)$

7) A function $f(x)$ will be increasing (decreasing) in R if it is so in every interval of R.