1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 03 (Class XII, Semester - 1)
Module Name/Title	Application of Derivatives – Part 1
Module Id	lemh_10601
Pre-requisites	Variables, Change in quantities,
Objectives	After going through this lesson, the learners will be able to understand the following: • Rate of change of quantities • Derivative as rate measure • Related Rates
Keywords	Rate of change of quantities, Related rate, Derivative

2. Development Team

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1. Introduction:

You have already learnt to differentiate a function with respect to a variable and also with respect to another function. You learnt how to find derivative of composite functions, inverse trigonometric functions, implicit functions, exponential functions and logarithmic functions etc. What do we actually mean by differentiation? Differentiation is the process of finding the difference by growth of a dependent variable or function with respect to the growth of related independent variable. Whenever you want to know how quickly one quantity is changing with respect to another quantity, you are basically required to find a derivative. It is the rate of change of one quantity with respect of another quantity.

You are familiar that to know how the position of a moving car

changes with progress of time, you are required to calculate, the derivative $\frac{ds}{dt}$, where *s* represents the distance travelled by the car in time *t*.

Derivatives have wide range of applications in various disciplines in real life situations like, engineering, sciences, social sciences, economics and in many other fields. Even it is used in history for predicting the life of a stone, in geography we use it to study the gases present in the atmosphere. Pilots use it to measure the pressure in the air. In ancient times, shipwrecks used to occur because the ship was not where the captain thought it should be, because to know velocity and acceleration we need derivatives. Even government uses derivatives for population census.

Hence without use of derivatives no technology development of the society would have been possible.

In following modules, under the title 'Application of Derivatives',

we will learn how derivatives can be used

- to determine rate of change of various quantities,
- to find the equations of tangent and normal to a curve at a point,
- to find intervals in which a function is increasing or decreasing,
- to find turning points on the graph of a function which in turn will help us to locate points at which largest or smallest value (locally) of a function occurs,
- to solve practical problems on maxima and minima,
- to find approximate value of certain quantity and so on.

2. Rate of change of quantities

Let, $y = f(x)$ be a function of x and Δy be the change in y corresponding to a small change Δx in x, then $\frac{\Delta y}{\Delta x}$ represents the change in y due to a unit change in x. We say that $\frac{\Delta y}{\Delta x}$ represents the average rate of change of y with respect to x, as x changes from x to $x+\Delta x$.

Take $\Delta x \to 0$, then limiting value of $\frac{\Delta y}{\Delta x}$ in the interval $[x, x+\Delta x]$ will become instantaneous rate of change of y with respect to x.

Hence,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \text{Instantaneous rate of change of y with respect to } x$$

We know that,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \text{Instantaneous rate of change of y with respect to } x.$$

Generally the word 'Instantaneous' is not used, hence, $\frac{dy}{dx}$ which is also written as $f'(x)$ represents the rate of change of y with respect to x for a definite value of x.

And, $\left[\frac{dy}{dx}\right]_{x=x_0}$ or $f'(x_0)$represents the rate of change of y with respect to x at $x = x_0$.

If we have two variables x and y varying with respect to another variable t, such that

$x = f(t)$ and $y = g(t)$, then by chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} x \frac{du}{dx}, \quad \text{if } \frac{dx}{dt} \neq 0$$

Thus, the rate of change of y with respect to x can be calculated using the rate of change of y and that of x both with respect to t.

So we can find rate of change of any dependent quantity with respect to the change in a quantity related to it.

In all the modules under the title 'Application of Derivatives' the term "rate of change" would mean the instantaneous rate of change unless stated otherwise.

3. Derivative as rate measure

The derivative of a function (or dependent variable) with respect to the independent variable can be used as a rate-measure of the function (or dependent variable) per unit change in the independent variable.

In the previous discussion,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

physically represents the rate-measure of y with respect to x.

thus, $\left[\frac{dy}{dx}\right]_{x=x_0}$ or $f'(x_0)$ represents the rate-measure of y with respect to x at $x = x_o$. If y increases as the independent variable x increases then the sign of $\frac{dy}{dx}$is positive and if y decreases as x increases then the sign of $\frac{dy}{dx}$is negative.

We know that total cost of production of any commodity in a factory will vary with the number of items produced. Hence number of items produced will be an independent variable and total cost of production will be dependent variable.

Marginal Cost (MC) is the instantaneous rate of change of total cost with respect to the number of items produced at an instant and Marginal Revenue (MR) is the instantaneous rate of change of total revenue with respect to the number of items sold at an instant.
Let us consider certain examples to understand application of "Derivatives" in different situations;

Example 1:
Find the rate of change of the area of a circle with respect to its radius r when r = 4 cm.
Solution:
The area A of a circle with radius r is given by,

$$A = \pi r^2.$$

Therefore, the rate of change of the area A with respect to its radius r will be obtained by differentiating 'A' with respect to r and is given by,

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2)$$

$$= 2\pi r$$

When r = 4 cm,

$$\left[\frac{dA}{dt}\right]_{r=4} = 2\pi(4) = 8\pi$$

Thus, the area of the circle is changing at the rate of 8π cm^2/cm.

Example 2:
Find the rate of change of the volume of a sphere with respect to its diameter when its diameter is 6 metre.
Solution:
Let *V* be the volume, *D* the diameter and *r* be the radius of the sphere,
Then, volume of the sphere

$$V = \frac{4}{3}\pi r^3$$

and $D = 2r$

$$\therefore V = \frac{4}{3}\pi\left[\frac{D}{2}\right]^3$$

$$= \frac{4}{24}\pi[D]^3 = \frac{1}{6}\pi[D]^3$$

Hence, differentiating *V* with respect to *D*, Rate of change of volume of the sphere with respect to its diameter

$$\frac{\mathrm{dV}}{\mathrm{dD}} = \frac{3}{6}\pi[D]^2 = \frac{1}{2}\pi[D]^2$$

$$= \frac{1}{2}\pi(6)^2 = 18\pi \ \ \mathrm{m^3/m}$$

Example 3:

Determine the rate of change of volume of a variable balloon with respect to the variable *x*, if the sphere always remains spherical and its diameter 'd' at any point of time is given by,

$$d = \frac{3}{2}(2x + 3)$$

Solution:

Let the volume and the radius of the sphere be by *V* and *r* respectively at any point of time,,

Then, volume of the sphere

$$V = \frac{4}{3}\pi r^3$$

and $$d = 2r = \frac{3}{2}(2x + 3)$$

$$\Rightarrow \ r = \frac{3}{4}(2x + 3)$$

$$V = \frac{4}{3}\pi[2x + 3]^3$$

$$= \frac{9}{16}\pi[2x + 3]^3$$

Differentiating w.r.t. *x* we get,

$$\frac{\mathrm{dV}}{\mathrm{dx}} = \frac{9\pi}{16}.3.(2x + 3)^2.\frac{d}{\mathrm{dx}}(2x + 3)$$

$$\frac{27\pi}{8}.(2x + 3)^2$$

Hence, rate of change of volume of the balloon with respect to the variable *x*, at any point of time is given by

$$\frac{\mathrm{dV}}{\mathrm{dx}} = \frac{27\pi}{8}.(2x + 3)^2$$

Example 4:

The length *x* of a rectangle is decreasing at the rate of 5 cm /minute and the width *y* is increasing at the rate of 4 cm /minute. When *x* = 8 cm and *y* = 6 cm, find the rate of change of perimeter and area of the rectangle.

Solution:

Given that,

$$\frac{dx}{dt} = -5 \text{ cm /minute}, \qquad \frac{dy}{dt} = 4 \text{ cm /minute} \quad \ldots\ldots\ldots(i)$$

(i) Let P be the perimeter of the rectangle at any instant, then

$$P = 2(x + y)$$

Differentiating both the sides with respect to time t, we get

$$\frac{dP}{dt} = 2 \; [+] \qquad \ldots\ldots\ldots(ii)$$

From (i) and (ii),

$$\frac{dP}{dt} = 2(-5 + 4) = -2 \text{ cm /minute}$$

The perimeter of the rectangle is decreasing at the rate 2 cm /minute.

(ii) Let A be the area of the rectangle at any instant, then

$$A = x.y$$

Differentiating both the sides with respect to time t, we get

$$\frac{dA}{dt} = \frac{dx}{dt}.y + x.\frac{dy}{dt}$$

$$= [-5.6 + 8.4] = 2 \text{ cm}^2 \text{ /minute}$$

Hence, Area of the rectangle is increasing at the rate 2 cm^2 /minute.

4. Related Rates :

Generally we come across with the problems when the rate of change of one quantity is required corresponding to the given rate of change of another quantity. For example, we may require rate of change of volume of a spherical balloon when rate of change of its radius is known. In such cases we are required to find relation connecting such quantities and differentiate them with respect to time. Let us consider certain examples to understand the procedure.

Example 5:

A stone is dropped into a quiet lake and waves move in circles. If radius of a circular wave increases at the rate of 4 cm/sec, find the rate of increase in its area at the instant when its radius is 10 cm.

Solution:

Let, at any instant t, the radius of the circle be r cm and its area be A cm^2. Since the radius of the circular wave is increasing at the rate of 4 cm/sec, therefore,

$$\frac{dr}{dt} = 4 \text{ cm/sec} \quad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Now, $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$

and $A = \pi r^2$

hence, $\frac{dA}{dt} = \frac{d(\pi r^2)}{dr} \cdot \frac{dr}{dt}$

$$= 2\pi r . 4 \quad \ldots\ldots\ldots \text{ using (i)}$$

$$= 8\pi r \text{ cm}^2\text{/sec}$$

$$\Rightarrow \left[\frac{dA}{dt}\right]_{r=10} = 80\pi \text{ cm}^2\text{/sec}$$

Thus area of the circle is increasing at the rate of (80π) cm^2/sec at the instant when radius is 10 cm.

Example 6:

Find the rate of change of surface area of a spherical balloon, if its volume is increasing at the rate of 20 cm^3/ sec at the instant when its radius is 8 cm.

Solution:

Let, at any instant of time *t*, *r* be the radius, *V* be the volume and *S* be the surface area of the balloon. Then according to the sum,

$$\frac{dV}{dt} = 20 \text{ cm}^3 \text{ /sec} \quad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Now, $V = \frac{4}{3}\pi r^3 \quad \ldots\ldots\ldots\ldots\ldots\text{(ii)}$

Differentiating with respect to *t*

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

Using (i) and (ii) we get,

$$20 = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) \cdot \frac{dr}{dt}$$

$$\Rightarrow \quad 20 = \left(\frac{4}{3}\pi . 3r^2\right) \cdot \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{5}{\pi r^2} \quad \ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

Surface area *S* of the balloon is given by,

$$S = 4\pi r^2$$

$$\Rightarrow \quad \frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$$

$$\frac{d(4\pi r^2)}{dr} \cdot \frac{dr}{dt}$$

$\frac{d(4\pi r^2)}{\text{dr}} \cdot \frac{5}{\pi r}$ from (iii)

$= 4\pi(2r)\frac{5}{\pi r^2}$

$\frac{\text{dS}}{\text{dt}} = \frac{40}{r}$ …………….(iv)

$\left[\frac{\text{dS}}{\text{dt}}\right]_{r=8} = \frac{40}{8}$

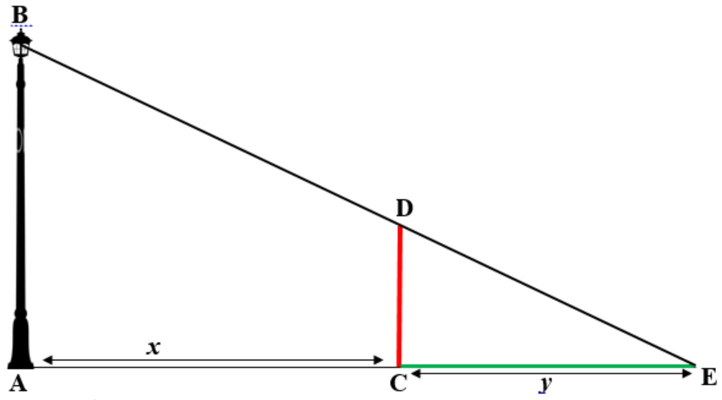
$= 5$ cm^2/sec

Hence, the rate of change of surface area of the spherical balloon is 5 cm^2/sec when its radius is 8 cm.

Example 7:

A man 2 metre high, walks at a uniform speed of 6 metre per minute away from a lamp post which is 5 metre high. Find the rate at which the length of his shadow increases.

Solution:



Let AB be the lamp post. The man CD be at a distance *x* metre from the lamp post and *y* metre be the length of his shadow at any time *t*, the shadow of the man is CE in the figure, then

AB = 5 metre , CD = 2 metre

and $\frac{\text{dx}}{\text{dt}} = 6$ metre / minute …………….(i)

Triangles ABE and CDE are similar,

$\therefore \quad \frac{AB}{CD} = \frac{AE}{CE}$

$\Rightarrow \quad \frac{5}{2} = \frac{x+y}{y}$

$\Rightarrow \quad 3y = 2x$

Differentiating with respect to *t*,

$$\Rightarrow \quad 3\frac{dy}{dt} = 2\frac{dx}{dt}$$

$$\Rightarrow \quad 3\frac{dy}{dt} = 2(6) \qquad \text{from (i)}$$

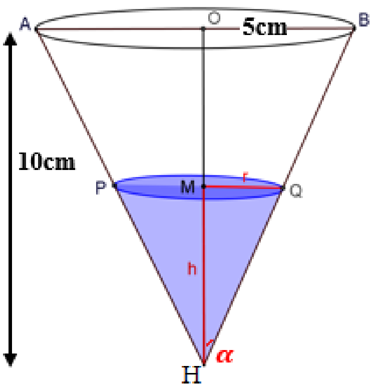
$$\Rightarrow \quad \frac{dy}{dt} = 4 \text{ metre / minute}$$

Thus the length of the shadow increases at the rate of 4 metre / minute.

Example 8:

Water is poured at the rate of 1.5 cm^3/minute into an inverted cone whose base radius is 5 cm and depth is 10 cm. Find the rate at which the water level in the cone is rising when the depth is 4 cm.

Solution:



From the figure,

the height of the cone HO = 10 cm and the radius OB = 5 cm.

Let semi-vertical angle of the cone be α, then

$$tan\alpha = \frac{5}{10} = \frac{1}{2} \qquad \text{................(i)}$$

$\frac{MQ}{MH}$

$$\Rightarrow \quad MQ = MHtan\alpha \qquad \text{................(ii)}$$

$$htan\alpha$$

Let *h* be height of water filled in the cone at time *t* and V be the volume of the water filled, then

$$V = \frac{1}{3}\pi(MQ)^2(MH)$$

$$\frac{1}{3}\pi(htan\alpha)^2(h)$$

$$\frac{1}{3}\pi h^3 tan^2\alpha$$

$$\frac{1}{3}\pi h^3 \left(\frac{1}{2}\right)^2$$ using (i) and (ii)

$$\frac{1}{12}\pi h^3$$

Differentiating with respect to time *t* we get,

$$\frac{dV}{dt}\ \frac{1}{12}\pi.\ 3h^2\,\frac{dh}{dt}$$

$$\frac{1}{4}\pi h^2\,\frac{dh}{dt} \qquad \ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

Water is poured at the rate of 1.5 cm^3/minute into the cone therefore volume is increasing at the rate of 1.5 cm^3/minute, hence,

$$\frac{dV}{dt} = 1.5 = \frac{3}{2} \qquad \ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

From (iii) and (iv)

$$\frac{3}{2}\frac{1}{4}\pi h^2\,\frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{6}{\pi h^2}$$

$$\left[\frac{dh}{dt}\right]_{h=4} = \frac{6}{\pi(4)^2} = \frac{3}{8\pi} \text{ cm/minute}$$

Hence the water level in the cone is rising at the rate $\frac{3}{8\pi}$ cm/minute when the depth is 4 cm.

Example 9:

The total cost $C(x)$ in Rupees, associated with the production of x units of an item is given by $C(x) = 0.002\,x^3 - 0.07\,x^2 + 35\,x + 6520$, Find the marginal cost when 5 units are produced, by marginal cost we mean the instantaneous rate of change of total cost at any level of output.

Solution:

The total cost $C(x)$ is given by

$$C(x) = 0.002\,x^3 - 0.07\,x^2 + 35\,x + 6520$$

Since marginal cost is the rate of change of total cost with respect to the output, we have

$$\text{Marginal cost (MC)} = \frac{dC}{dx}$$

$$= 0.002\,(3x^2) - 0.07\,(2x) + 35$$

$$= 0.006\,(x^2) - 0.14\,(x) + 35$$

Therefore, $\left(\frac{dC}{dx}\right)_{x=5} = 0.006\,(5^2) - 0.14\,(5) + 35$

$$= 0.006\,(25) - 0.14\,(5) + 35$$

$$= 0.150 - 0.7 + 35$$

$$= 34.45$$

Hence, the required marginal cost is Rupees 34.45 when 5 units are produced.

Example 10:

The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 13\,x^2 + 26\,x + 15$. Find the marginal revenue, when $x = 5$, where by marginal revenue we mean the rate of change of total revenue with respect to the number of items sold at an instant.

Solution:

The total revenue is given by

$$R(x) = 13\,x^2 + 26\,x + 15$$

Since marginal revenue is the rate of change of total revenue with respect to the number of items sold, we have

$$\text{Marginal Revenue (MR)} = \frac{dR}{dx}$$

$$= 13(2x) + 26$$

$$= 26\,x + 26$$

Therefore, $\left(\frac{dR}{dx}\right)_{x=5} = 26\,(5) + 26$

$$= 130 + 26$$

$$= 156$$

Hence, the required marginal revenue is Rupees 156 when 5 items are sold.

5. Summary

1) If a quantity *y* varies with another quantity *x*, satisfying some rule,

$y = f(x)$, then $\frac{dy}{dx}$ which is also written as $f'(x)$ represents the rate of change of *y* with respect to *x* and $\left[\frac{dy}{dx}\right]_{x=x_0}$ or $f'(x_0)$ represents the rate of change of *y* with respect to *x* at $x = x_0$.

2) If we have two variables *x* and *y* varying with respect to another variable *t*, such that,

$x = f(t)$ and $y = g(t)$, then by chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} x \frac{du}{dx}, \quad \text{if } \frac{dx}{dt} \neq 0$$

3) The derivative of a function (or dependent variable) with respect to an independent variable can be used as a rate-measure of the function (or dependent variable) per unit change in the independent variable.

4) Marginal Cost (MC) is the instantaneous rate of change of total cost with respect to the number of items produced at an instant and Marginal Revenue (MR) is the instantaneous rate of change of total revenue with respect to the number of items sold at an instant.