1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 03 (Class XII, Semester – 1)
Module Name/Title	Continuity and Differentiability - Part 1
Module Id	lemh_10501
Pre-requisites	Knowledge about Continuity and Differentiability
Objectives	After going through this lesson, the learners will be able to understand the following: • Continuous Function • Properties of Continuous Function • Discontinuous Function • Kinds of Discontinuity
Keywords	Continuous function, Discontinuous function, Differentiable function.

2. Development Team

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1. Continuity of a Function in an Interval

Analytically, a real function is continuous at a point if the graph of the function has no break at that point.

One can deduce from the graph that the value of the function at nearby points (neighbouring points) on x- axis remain close to each other (including the point) without lifting the pen from the plane of the paper.

(i) Continuity of a Real Function at a Point

Let's understand the concept of continuity with some examples.

Example 1: Consider the function 'f' defined at every point of real line.

$$f(x) = \begin{cases} 2 & if x < 0 \\ 3 & if x > 0 \end{cases}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2 = 2$$

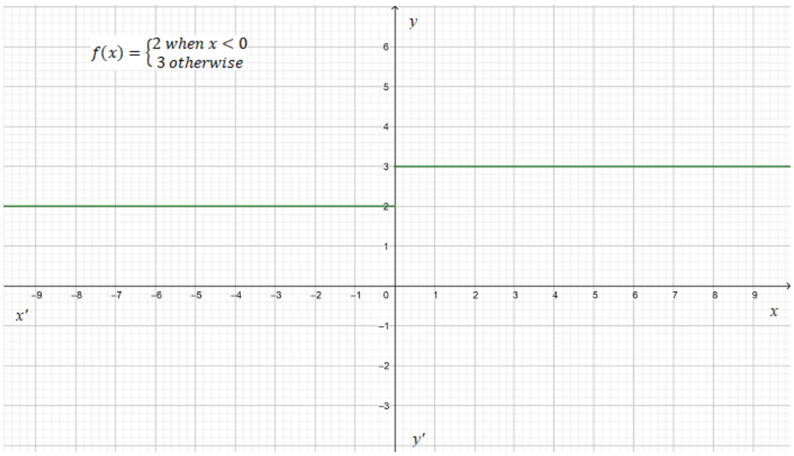
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 3 = 3$$

f(x) = 2 at x = 0

As, $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$, it implies that $\lim_{x\to 0} f(x)$ does not exist.

It can be observed while we try to draw the graph of the above function, we cannot draw it in one stroke i.e. without lifting pen from the plane of the paper. In fact, we need to lift off the pen when we come to zero from the left.

So function is not continuous at x=0.



Example 2: Consider the function

$$f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 1 & x = 0 \end{cases} \text{ at } x=0$$

Let's evaluate $\lim_{x \to 0} f(x)$ i.e., $\lim_{x \to 0} \frac{|x|}{x}$

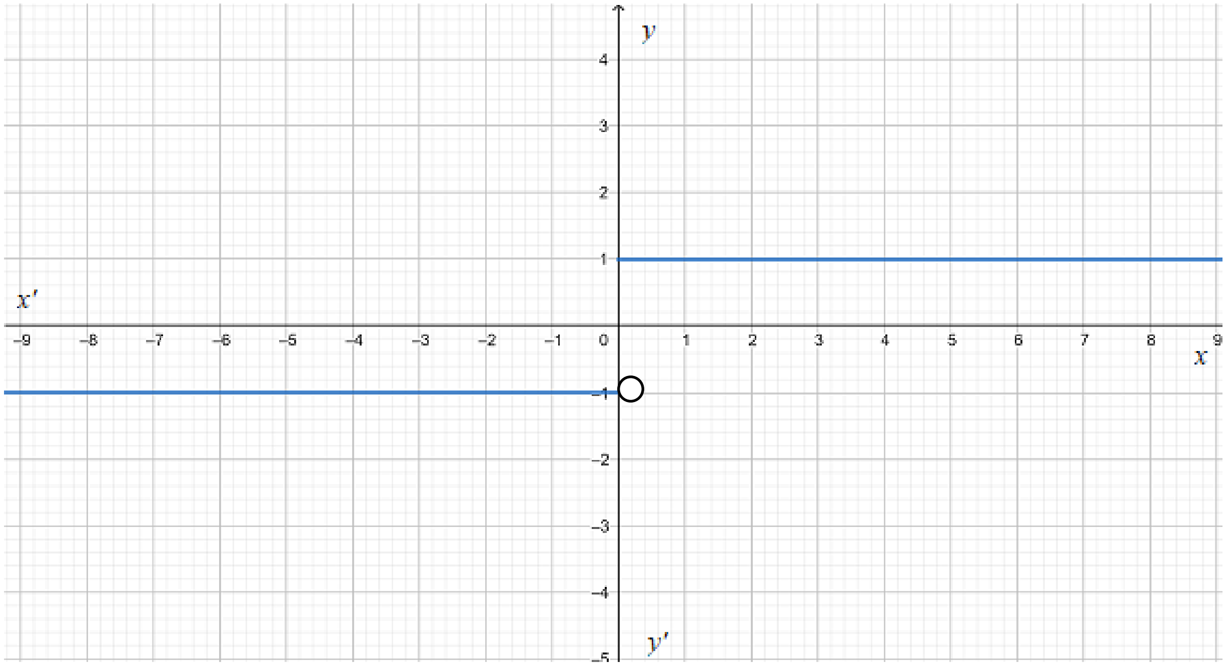
Left Hand Limit at x=0

$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} (\frac{-x}{x})$=-1(|x|= -x when x < 0)

Right Hand Limit at x=0

$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} (\frac{x}{x})$= 1(|x|= x when x > 0)

Thus, $\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$ it implies that $\lim_{x \to 0} f(x)$ does not exist.



(i) Discontinuous Function: A function ‘f’ is discontinuous at x=c if

1. If f is not defined at x=c i.e. f(c) does not exist.
2. f(x) does not exist if either

   (i) $\lim_{x \to c^-} f(x)$does not exist or

   (i) $\lim_{x \to c^+} f(x)$ does not exist.
3. Both $\lim_{x \to c^-} f(x)$and $\lim_{x \to c^+} f(x)$exist but are not equal.
4. $\lim_{x \to c} f(x)$ exists but $\lim_{x \to c} f(x) \neq$ f(c)
5. (i) $\lim_{x \to c^-} f(x)$exists but $\lim_{x \to c^-} f(x) \neq$ f(c)

   (ii) $\lim_{x \to c^+} f(x)$ exists $\lim_{x \to c^+} f(x) \neq$ f(c)

(ii) Continuity of a function in an Interval:

1. A function ‘f’ is continuous in an open interval (a,b) if and only if ‘f’ is continuous at every point of the interval (a,b); and
2. A function is said to be continuous in the closed interval [a,b] if

(i) 'f' is continuous in the open interval (a,b)

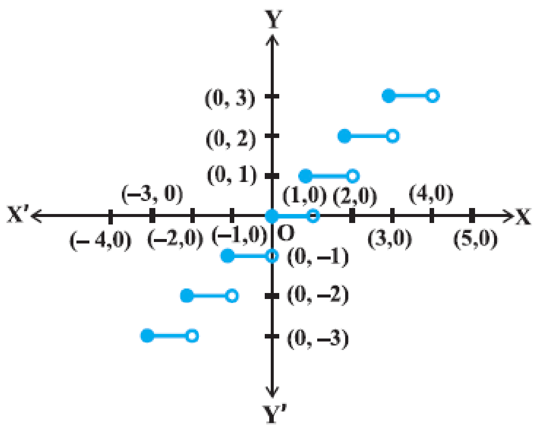
(ii) it is continuous at 'a' from the right and

(iii) it is continuous at 'b' from the left.

2. Continuous Function: A function is continuous function if and only if it is continuous at every point of its domain.

Example 1: Is [x], a greatest integer function less than or equal to x, is a continuous function on the set of real numbers?

Solution: [x], a greatest integer function less than or equal to x is a discontinuous function, as it is discontinuous at integral values, if we see its graph.



Mathematically, let's discuss the continuity of the function f(x) = [x] at x=c,

$$\lim_{x \to c^+} f(x) = \lim_{h \to 0} f(c+h) = [c+h] = c$$

(c< c+h < c+1 ∴ [c+h] =c)

$$\lim_{x \to c^-} f(x) = \lim_{h \to 0} f(c-h) = [c-h] = c-1$$

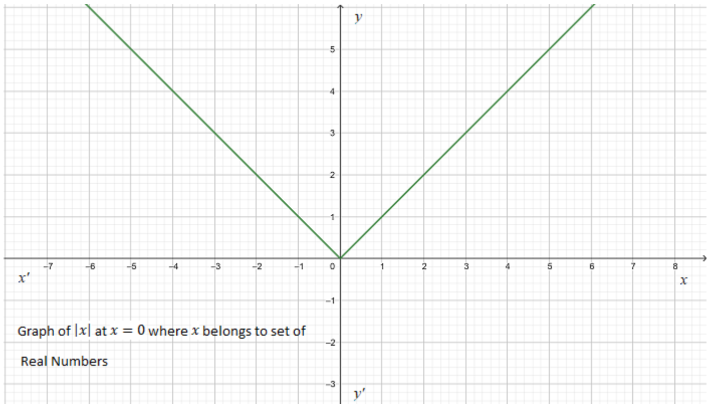
(c-1<c-h<c∴[c-h] =c-1)

Since LHL (left hand limit) is not equal toRHL (right hand limit), so$\lim_{x \to c}[x]$does not exit.

Therefore, f(x) =[x] is discontinuous function at x=c, i.e. at all integral values, hence becomes discontinuous on the set of real numbers.

Example 2: Is |x| a continuous function on the set of real numbers ?

Solution :$|x|$ is a continuous function at x=c, any real number on the set of real numbers, if we see its graph we can see the graph of the function is without break.



Mathematically, let's see the function's continuity at x=c, where 'c' is any real number.

(LHL at x=c).

|x| = |c-h| when x<0 (put x=c-h, where h is too small, where h→0.)

= c (RHL at x = c)

= |c+h| when x ≥ 0 (put x= c+h, where h is too small, where h→0.)

= c

Since, LHL = RHL (at x=c) therefore, at x=c, the function becomes continuous.

Hence, function f(x) = |x| is a continuous function at all values of x and hence continuous on the set of real numbers.

Properties of Continuous function:

Theorem 1: Let f and g be two real functions continuous at a real number c, then

(i) f + g is continuous at x=c

(ii) f – g is continuous at x=c

(iii) f-g is continuous at x=c

(iv) (f/g) is continuous at x= c.(provided g(c) ≠ 0)

Corollary: (i) αf is continuous at x= c for all α ε R.

(ii) If α = -1 then – f is continuous at x=c for all α ε R.

(iii) Continuity of f(x) also implies continuous 1/f provided f(c) ≠ 0.

Theorem 2: A polynomial function is continuous everywhere.

Solution: Recall that a function f(x) =a_0x^n + a_1x^{n-1} + a_2x^{n-2}+ ……………. + a_nx^0, is a polynomial function for some natural number 'n' and 'c' be any real number then

Let x→c f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2}+ ……………. + a_n= f(c).

Hence, by definition polynomial function is a continuous function as it is continuous at 'c' which is any real number, so it is continuous at every real number.

Corollary: 1) Every constant function is continuous.

2) Every identity function is a polynomial function, hence continuous.

Theorem 2: A rational function is continuous at every point of its domain.

Solution: Recall that every rational function 'f' is given by f(x) = $\frac{p(x)}{q(x)}$, q≠ 0 where 'p' and 'q' are polynomial functions. The domain of 'f' is all real numbers except points at which 'q' is zero. Since polynomial functions are continuous functions, hence rational function is a continuous function.

Theorem 3: If f is continuous at c, then |f| is also continuous at x=c.

Note: The absolute function of a function 'f' i.e. |f| is defined by |f| (x) = |f(x)| for all x ε R.

Corollary: 1) The converse of the above theorem may not be true.

Example: Let's consider the function f(x) = $\begin{cases} 3 \text{ if x is an integer} \\ -3 \text{ if x is not an integer} \end{cases}$ where D_f = R.

Here, |f(x)| is continuous function but f(x) is not continuous function. Think why?

Theorem 4: If function is one-one ,onto and continuous function on domain [a,b] and range [c,d] then the inverse of function i.e.

f^{-1}: [c,d] →[a,b] is continuous on [c,d].

Theorem 5: If f is continuous at 'c', g is continuous at f(c) and gof is defined, then gof is continuous at 'c'.

Example: |1-x+|x|| is a continuous function.

Solution: (i) 1-x, being a polynomial function is a continuous function.

(ii) |x| is also a continuous function

(iii) So 1-x + |x| is also continuous function as sum of two continuous function is continuous.

(iv) Now let us take f(x) = 1-x +|x| and g(x) = |x|

Then, (gof) (x) = g[f(x)] = |f(x)|=|1-x+|x||.

Now if 'f' and 'g' are two continuous functions then their composition (gof) (x) is also a continuous function.

Therefore, |1-x+|x|| is a continuous function.

Theorem 6: All the basic trigonometric functions i.e. sinx, cosx, tanx, cotx, secx and cosec x within its respective domain are continuous function.

Proof: Let's see one of the function f(x) = sinx.

Here, Domain of sin x is real number R.

∴f(c) = sin c where c ε R.

Now, $\lim_{x\to 0} f(x) = \lim_{h\to 0} f(c+h)$

$= \lim_{h\to 0} \sin(c+h)$

$= \lim_{h\to 0} (\sin c \cos h + \cos c \sin h)$

$= \sin c . \lim_{h\to 0} \cos h + \cos c . \lim_{h\to 0} \sin h$

= sin c.(1) + cos c.(0)

= sin c = f(c)

So f (c) $= \lim_{x\to c} f(x)$ = sin c, hence sine function f(x) = sinx is continuous function for every x ε R.

Similarly we can see proof for other trigonometric functions.

Theorem: The basic inverse trigonometric functions i.e. $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\mathrm{cosec}^{-1} x$ and $\sec^{-1} x$ are continuous functions in their respective domain.

Summary

- A real valued function is continuous at a point in its domain if the limit of function at that point equals the value of function at that point.
- A function is continuous function if and only if it is continuous at every point of its domain.
- Let f and g be two real functions continuous at a real number c, then
  - (i) f + g is continuous at x=c
  - (ii) f – g is continuous at x=c
  - (iii)f-g is continuous at x=c
  - (iv)(f/g) is continuous at x= c.(provided g(c) ≠ 0)
- If f is continuous at ‘c‘, g is continuous at f(c) and the composite function gof is defined, then gof is continuous at c.
- All the basic trigonometric functions are continuous function within its respective domain.
- The basic inverse trigonometric functions are continuous functions in their respective domain.