1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 03 (Class XII, Semester - 1)
Module Name/Title	Determinant - Part 4
Module Id	lemh_10404
Pre-requisites	Basic knowledge about Adjoint of a Square Matrix
Objectives	After going through this lesson, the learners will be able to understand the following: • Adjoint of a Square Matrix • Reversal law • Inverse of a matrix
Keywords	Adjoint of a Square Matrix, Reversal Law, Matrix Inverse

2. Development Team

Role	Name	Affiliation
National MOOC Coordinators (NMC)	Prof. Amarendra P. Behera	CIET, NCERT, New Delhi
Program Coordinator	Dr. Indu Kumar	CIET, NCERT, New Delhi
Course Coordinator	Prof. Til Prasad Sharma	DESM, NCERT, New Delhi
Subject Coordinator	Anjali Khurana	CIET, NCERT, New Delhi
Subject Matter Expert (SME)	Dr. Monika Sharma	Shiv Nadar University, Noida
Revised By	Manpreet Kaur Bhatiya	IITNM College, GGSIP University
Review Team	Prof. Bhim Prakash Sarrah Prof. V.P. Singh (Retd.) Prof. SKS Gautam (Retd.)	Assam University, Tezpur DESM, NCERT, New Delhi DESM, NCERT, New Delhi

TABLE OF CONTENTS

1. Adjoint of a Square Matrix

Let A=[A_{ij}] be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A. Then the transpose of the matrix of cofactors of elements of A is called the Adjoint of A and is denoted by Adj A.

Thus, Adj A = $[C_{ij}]^T \rightarrow (\text{adj } A)_{ij} = C_{ji}$ = Cofactor of a_{ji} in A.

If A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then,

Adj A=$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$,

where C_{ij} denotes the cofactor of a_{ij} in A.

Example: Find the Adjoint of matrix A=[a_{ij}]=$\begin{bmatrix} p & q \\ r & s \end{bmatrix}$

Solution: We have, Cofactor of a_{11}=s, Cofactor of a_{12} = -r, Cofactor of a_{21} = -q and, Cofactor of a_{22}=p

∴ Adj A =$\begin{bmatrix} s & -r \\ -q & p \end{bmatrix}^T = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$

Note: It is evident from this example that the Adjoint of a square matrix of order 2 can be easily obtained by interchanging the diagonal elements and changing signs of off-diagonal elements.

If A=$\begin{bmatrix} -2 & 3 \\ -5 & 4 \end{bmatrix}$, then by the above rule, we obtain

Adj A =$\begin{bmatrix} 4 & -3 \\ 5 & -2 \end{bmatrix}$

Example: Find the Adjoint of matrix A = $[a_{ij}]$ = $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & 3 \end{bmatrix}$

Solution: Let C_{ij} be cofactor of a_{ij} in A. Then, the cofactors of elements of A are given by

$C_{11} = \begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = 9$, $C_{12} = -\begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = -3$, $C_{13} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5$

$C_{21} = -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -1$, $C_{22} = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 4$, $C_{23} = -\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -3$,

$C_{31} = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -4$, $C_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = 5$, $C_{33} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1$

Adjoint of A is transpose of the matrix of cofactor matrix associated with A.

$$\text{adj } A = \begin{bmatrix} 9 & -3 & 5 \\ -1 & 4 & -3 \\ -4 & 5 & -1 \end{bmatrix}^T = \begin{bmatrix} 9 & -1 & -4 \\ -3 & 4 & 5 \\ 5 & -3 & -1 \end{bmatrix}$$

Theorem : Let A be a square matrix of order n. Then, A(adj A) = |A| I_n = (adj A)A.

Verification : If A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3, then,

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix},$$

where C_{ij} denotes the cofactor of a_{ij} in A.

$$A\,(\text{adj } A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

Since , $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ and this is true for the sum of products of the elements of a row (or column) with their corresponding cofactors

$$= |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

= |A| I_3

Similarly(adj A) A = |A| I_3=A(adj A)

Example. Compute the adjoint of the matrix A given by A = $\begin{bmatrix} 1 & 4 & 5 \\ 3 & 2 & 6 \\ 0 & 1 & 0 \end{bmatrix}$ and verify that

A(adj A) = |A|I=(adj A).

Solution. We have,

|A|=$\begin{bmatrix} 1 & 4 & 5 \\ 3 & 2 & 6 \\ 0 & 1 & 0 \end{bmatrix}$=1(0-6)-4(0-0)+5(3-0)=9

Let C_{ij} be cofactor of a_{ij} in A. Then, the cofactors of elements of A are given by

$C_{11} = \begin{vmatrix} 2 & 6 \\ 1 & 0 \end{vmatrix} = -6$, C_{12}=-$\begin{vmatrix} 3 & 6 \\ 0 & 0 \end{vmatrix} = 0$, C_{13}=$\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} = 3$,

C_{21}=- $\begin{vmatrix} 4 & 5 \\ 1 & 0 \end{vmatrix} = 5$,C_{22} =$\begin{vmatrix} 1 & 5 \\ 0 & 0 \end{vmatrix} = 0$, C_{23} = -$\begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} = -1$

$C_{31} = \begin{vmatrix} 4 & 5 \\ 2 & 6 \end{vmatrix} = 14$,$C_{32} = -\begin{vmatrix} 1 & 5 \\ 3 & 6 \end{vmatrix} = 9$, $C_{33} = \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} = -10$

∴ Adjoint of A is transpose of the matrix of cofactor matrix associated with A.

$$\text{adj } A = \begin{bmatrix} 9 & -3 & 5 \\ -1 & 4 & -3 \\ -4 & 5 & -1 \end{bmatrix}^T = \begin{bmatrix} 9 & -1 & -4 \\ -3 & 4 & 5 \\ 5 & -3 & -1 \end{bmatrix}$$

2. SINGULAR AND NON-SINGULAR MATRIX

Definition : A square matrix A is said to be singular if |A| = 0

Example: The determinant $\begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix}$ is 1 x 8 - 2 x 4 = 0,

Hence A is singular matrix.

Definition : A square matrix A is said to be non-singular if |A| ≠ 0.

Let A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then |A| = $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$= 4-6 =-2 ≠ 0

Hence A is a non-singular matrix.

Example: For what value of x the matrix A =$\begin{bmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{bmatrix}$ is singular?

Solution: The matrix A is singular, if$|x|$ = 0

$$\begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{vmatrix} = 0$$

On expanding along first row, we get

$$1\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} -(-2)\begin{vmatrix} 1 & 1 \\ x & -3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ x & 2 \end{vmatrix} = 0$$

Again simplifying, we get

(-6-2) + 2 (-3-x)+3(2-2x)=0

-8-6-2x+6-6x=0

-8x-8=0

x =-1

Example : If A is non-singular matrix of order 3 , then | adj A| = $|A|^2$

Solution : Since A is non-singular matrix of order three , then |A| $\neq 0$

We know that A(adj A) = $|A|I_3$ = (adj A)A.

$$\Rightarrow A(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$\Rightarrow |A(\text{adj } A)| = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$$

⇒ |A ||(adj A)|= $|A|^3$

⇒ |(adj A)|= $|A|^2$

In fact , the above result is true for any non-singular matrix A of order n.

In general, if A is a non-singular matrix of order n, then |adj(A)|=$|A|^{n-1}$.

Example : If A is an non-singular matrix of order 3 and |A|=5 , then find |adjA| .

Solution : Here A is an non-singular matrix of order 3.

Therefore ,| adj A| = $|A|^2$

|adj A| = $|A|^2$ by |(adj A)|= $|A|^{n-1}$

(|A| =5)

⇒ |adj A| = 5^2= 25

Theorem : If A and B are nonsingular matrices of the same order, then AB and BA are also non-singular matrices of the same order.

Theorem : The determinant of the product of matrices is equal to product of their respective determinants, that is |AB|=|A||B|, where A and B are square matrices of the same order.

3. INVERSE OF MATRIX

Inverse: A non-singular square matrix of order n is invertible if there exists a square matrix B of the same order such that $AB=I_n=BA$.

In such a case, we say that the inverse of A is B and we write, $A^{-1}=B$.

Theorem : **A square matrix A is invertible if and only if A is non-singular matrix. The inverse of matrix A is then given by** $A^{-1} = \frac{adjA}{|A|}$

Proof : Let A be a square matrix of order n.

First , let A be invertible , then there exists a square matrix B of order n such that

$AB = I_n = BA$

$\Rightarrow |AB| = |I_n|$

$|A|\,|B| = 1$

$\Rightarrow |A| \neq 0$

$\Rightarrow$A is non-singular .

Conversely , let A be non-singular, i.e. $|A| \neq 0$

$A\,(adj\,A) = |A|\,I_n = (adj\,A)A$

$A\left(\frac{1}{|A|}adjA\right) = \left(\frac{1}{|A|}adjA\right)A$ (As $|A| \neq 0$)

$\Rightarrow AB = I_n = BA$ where $B = \frac{1}{|A|}adjA$

Therefore , A is invertible.

And inverse of A is given by $A^{-1} = \frac{adjA}{|A|}$

Example: Compute the inverse of the matrix A given by A = $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$.

Solution: Firstly we evaluate the determinant of the matrix

$|A|= 1(16-9)-3(4-3)+3(3-4) = 1 \neq 0$, so inverse exists.

$A^{-1} = \frac{adjA}{|A|}$

Let C_{ij} be cofactor of a_{ij} in A. Then, the cofactors of elements of A are given by

$C_{11} = \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 7$, $C_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$, $C_{13} = \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = -1$

$C_{21} = -\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = -3$, $C_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$, $C_{23} = -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$,

$C_{31} = \begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = -3$, $C_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$, $C_{33} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$

So, the cofactor matrix is $C_{ij} = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

Adj A = ${C_{ij}}^T = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Therefore, $A^{-1} = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

The Inverse of a Matrix when it satisfies some Matrix Equation f(A)=0.

Example: Show that A= $\begin{bmatrix} 4 & -3 \\ 5 & -2 \end{bmatrix}$satisfies the equation $A^2 -6A + 17I =0$. Hence, find A^{-1}.

Solution : Here, A= $\begin{bmatrix} 4 & -3 \\ 5 & -2 \end{bmatrix}$

Therefore ,A^2= AA = $\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4-9 & -6-12 \\ 6+12 & -9+16 \end{bmatrix} = \begin{bmatrix} -5 & -18 \\ 18 & 7 \end{bmatrix}$

-6A = (-6) $\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -12 & 18 \\ -18 & -24 \end{bmatrix}$

And 17I = 17 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix}$

Therefore

A^2 -6A + 17 I_2 = $\begin{bmatrix} -5-12+17 & -18+18+0 \\ 18-18+0 & 7-24+17 \end{bmatrix}$ = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$=0

Thus, the matrix A satisfies the equation x^2-6x +17 =0

Now A^2 -6A + 17 I_2=0

Which implies A^2 -6A =- 17 I_2

$A^{-1}(A^2 -6A) = A^{-1}(- 17 I_2)$ (Pre-multiplying both sides by A^{-1})

$A^{-1}A^2$ -6 $A^{-1}A$ = - 17 A^{-1} I_2

A-6I_2= - 17 A^{-1}

A^{-1} =-$\frac{1}{17}$ (A-6I_2)

A^{-1} =$\frac{1}{17}$ (6I_2 -A)

=$\frac{1}{17}${ $\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$ –$\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$}

=$\frac{1}{17}\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}$

To Solve Matrix Equations :

Find the matrix X for which $\begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix}$X = $\begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$

Solution : Let P= $\begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix}$and Q= $\begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$. Then the given matrix equation is PX=Q.

Therefore, |P| = $\begin{vmatrix} 1 & -4 \\ 3 & -2 \end{vmatrix}$= - 2+ 12 =10 $\neq$ 0.

So, P is an invertible matrix. Let C_{ij} be cofactors of a_{ij} in P=[a_{ij}].

Therefore , C_{11}= -2,C_{12}−3

C_{21}4 and C_{22} =1

Therefore , adj P = $\begin{bmatrix} -2 & -3 \\ 4 & 1 \end{bmatrix}^T$

= $\begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$

Therefore P^{-1} = $\frac{1}{|P|}$adj P = $\frac{1}{10}\begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$

Now PX= Q

Which implies P^{-1} (PX)= P^{-1} Q

(P^{-1} P)X= P^{-1} Q

I X= P^{-1} Q

⇒ X= P^{-1} Q

⇒ X= $\frac{1}{10}\begin{bmatrix}-2 & 4\\ -3 & 1\end{bmatrix}\begin{bmatrix}-16 & -6\\ 7 & 2\end{bmatrix}$

=$\frac{1}{10}\begin{bmatrix}32+28 & 12+8\\ 48+7 & 18+2\end{bmatrix}$

=$\begin{bmatrix}6 & 2\\ \frac{11}{2} & 2\end{bmatrix}$

Example : If A = $\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$, find (adj A)$^{-1}$

Solution : We have , A = $\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$

Therefore, |A| = 2(4-1) + 1(-2 +1) + 1(1-2) = 4 ≠ 0

On re-arranging the formula of A^{-1} we obtain (adj A)$^{-1}$= A/|A|

Therefore (adj A)$^{-1}$ = $\frac{1}{4}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$

Summary:

- If A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then, Adj A = $\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$, where C_{ij} denotes the cofactor of a_{ij}.
- A(adj A) = |A|I_n = (adj A)A where A is a square matrix of order n
- A square matrix A is said to be singular if |A| = 0.
- A square matrix A is said to be non- singular if |A| ≠ 0.
- If A is a non-singular matrix of order n, then $|adj(A)|=|A|^{n-1}$.
- If AB=I_n=BA where B is a square matrix , then B is called the inverse of A and we write, A^{-1} =B
- $(A^{-1})^{-1}$ = A
- A square matrix has an inverse if and only if it is non-singular.
- $A^{-1} = \frac{\text{adj}A}{|A|}$