1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 03 (Class XII, Semester - 1)
Module Name/Title	Determinant - Part 3
Module Id	lemh_10403
Pre-requisites	Basic knowledge about Applications of Determinants to Coordinate Geometry
Objectives	After going through this lesson, the learners will be able to understand the following: ● Minors and Cofactors ● Area of Triangle ● Collinear points
Keywords	Minors, Cofactors , Area of a Triangle, collinear points

2. Development Team

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TABLE OF CONTENT:

1. MINORS

Definition: Let A = $|a_{ij}|$ be a determinant of order n. The minor M_{ij} of a_{ij} in A is the determinant of order (n-1) obtained by leaving i^{th} row and j^{th} column of A.

Let's understand the Minor with an example.

Example : If A=$\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$

then

minor M_{11} of a_{11} = 5 , minor M_{12} of a_{12} = 4,

minor M_{21} of a_{21} = 2 , minor M_{22} of a_{22} = 1

Example: If A = $\begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ 2 & 2 & -3 \end{vmatrix}$

then let's find out minors of their elements.

Minor M_{11} of a_{11} = Determinant 2 x 2 obtained by leaving first row and first column of A.

Minor M_{11} of $a_{11} = \begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} = -6-2 = -8$

Similarly, we obtain other minors

Minor M_{12} of $a_{12} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -3-2 = -5$

Minor M_{13} of $a_{13} = \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 2-4 = -2$

Minor M_{21} of $a_{21} = \begin{vmatrix} -2 & 3 \\ 2 & -3 \end{vmatrix} = 6-6 = 0$

Exercise:

Similarly find Minor M_{22}, Minor M_{23}, Minor M_{31}, Minor M_{32}, Minor M_{33}.

2. COFACTORS

Definition :Let A = $|a_{ij}|$ be a determinant of order n. The cofactor C_{ij} of a_{ij} in A is $(-1)^{i+j}$ times the determinant of order (n-1) obtained by leaving i^{th} row and j^{th} column of A.

Therefore,

The cofactor C_{ij} of a_{ij} in A = $(-1)^{i+j} M_{ij}$, where M_{ij} is Minor of a_{ij} in A.

Thus, we have, $C_{ij} = M_{ij}$, if i+j is even and

$C_{ij} = -M_{ij}$, if i+j is odd.

Let's explore the cofactor through an example

Example: If A= $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$, then

Cofactor C_{11} of $a_{11} = (-1)^{1+1} M_{11} = M_{11} = 5$, Cofactor C_{12} of a_{12}= $(-1)^{1+2}$ M_{12} = $-M_{12}$= -4,

CofactorC_{21} of a_{21}=$(-1)^{2+1}$ M_{21} =$-M_{21}$= -2 ,

Cofactor C_{22} of a_{22}= $(-1)^{2+2}$ M_{22} = M_{22} =1.

Note: Cofactor C_{ij} = $-M_{ij}$, if i+j is odd and

C_{ij}= M_{ij}, if i+j is even.

Example: If A=$\begin{bmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ 2 & 2 & -3 \end{bmatrix}$,

then let's find out cofactors of some of its elements.

Cofactors C_{11} of a_{11}= $(-1)^{1+1}M_{11}$ = M_{11} =$\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix}$= -8

Cofactors C_{12} ofa_{12}= $(-1)^{1+2}M_{12}$ = $-M_{12}$= -$\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}$**= 5**

Cofactors C_{13}ofa_{13}= $(-1)^{1+3}M_{13}$ = M_{13}=$\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}$**= -2**

Cofactors C_{21} of a_{21}= $(-1)^{2+1}M_{21}$ = $-M_{21}$= $-\begin{vmatrix} -2 & 3 \\ 2 & -3 \end{vmatrix}$=0

Similarly, one can find cofactors of other elements .

Exercise:

Find cofactors C_{22}, C_{23}, C_{31}, C_{32} and C_{33} .

Note:

1) The value of determinant A is obtained by the sum of products of elements of a row (or a column) with corresponding cofactors.

   For example, $|A|$ = $a_{11}C_{11}$ +$a_{12}C_{12}$+$a_{13}C_{13}$.

   (Recall that we have expanded the determinant in Module 1 using the same technique)

2) If elements of a row (or column) are multiplied with the cofactors of any other row (or column), then their sum is zero

   For example, $a_{11}C_{21}$ +$a_{12}C_{22}$+$a_{13}C_{23}$= 0

3. APPLICATIONS OF DETERMINANTS TO COORDINATE GEOMETRY:

- **Area of a Triangle**

We know from the knowledge of co-ordinate geometry that the area of a triangle with vertices (x_1, y_1), (x_2,y_2), (x_3,y_3) is given by the expression

$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$.

This expression can also be written in the form of determinant as det = $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

Therefore ,the area of triangle ABC = the absolute value of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Example: Find the area of the triangle whose vertices are (3,8),(-4,2) and (5,1).

Solution: The area of a triangle is given by the absolute value of $\frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$

= $\frac{1}{2}$ [3 (2-1) -8(-4-5)+ 1 (-4-10)]

= $\frac{1}{2}$ (3+ 72 -14)= $\frac{61}{2}$ sq. units

Example: Using determinants, find the area of the triangle whose vertices are (-3,5), (7,2) and (3,-6).

Solution: The area of a triangle with vertices (-3,5),(3,-6) and (7,2) is given by the absolute value of $\frac{1}{2}\begin{vmatrix} -3 & 5 & 1 \\ 7 & 2 & 1 \\ 3 & -6 & 1 \end{vmatrix}$

Δ = $\frac{1}{2}$ [(-3) (2+6)- 5 (7-3)+1(-42-6)]

=$\frac{1}{2}$ (-24-20-48)

= $\frac{1}{2}$ (-92) = -46

Area = $|-46|$ = 46 sq. units

- **Condition of collinear of three points:**

Let the three points be A(x_1, y_1) , B(x_2, y_2) and C(x_3, y_3). The given points A, B and C will be collinear if and only if area of $\Delta ABC = 0$.

Which implies area of triangle formed by three collinear points A,B,C is zero if and only if

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Example: Prove that the points P (a, b+c), Q (b, c+a) and R (c, a+b) are collinear.

Solution . The given points P, Q and R are collinear if

$$\Delta=\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}=0$$

$$\Rightarrow\Delta=\begin{vmatrix} a & a+b+c & 1 \\ b & b+c+a & 1 \\ c & c+a+b & 1 \end{vmatrix} \qquad \text{(operating } C_2 \to C_2+C_1\text{)}$$

$$\Rightarrow\Delta=(a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} \qquad \text{(Taking common (a+b+c) from } C_2\text{)}$$

$\Rightarrow \Delta= (a+b+c) \times 0$ (because C_2, C_3 are identical)

$\Rightarrow \Delta= 0$

So, the points P (a, b+c), Q (b, c+a) and R (c, a+b) are collinear.

Example: If the points (2,-3), (a, -1) and (0,4) are collinear, find the value of a.

Solution: If given points are collinear, the

$$\begin{vmatrix} 2 & -3 & 1 \\ a & -1 & 1 \\ 0 & 4 & 1 \end{vmatrix}=0$$

Applying $R_2 \to R_2 - R_1$

$$\begin{vmatrix} 2 & -3 & 1 \\ a-2 & 2 & 0 \\ -2 & 7 & 0 \end{vmatrix}=0 \qquad \text{expanding the determinant along } C_3.$$

$$\begin{vmatrix} a-2 & 2 \\ -2 & 7 \end{vmatrix}=0$$

$7a -14+4=0$

$a=\frac{10}{7}$.

(C).Equation of a line passing through two given points

Let the two points be A(x_1, y_1) and B(x_2, y_2). Let P(x, y) be any point on the line joining A and B.

Then, points P, A and B are collinear. Therefore,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$$

This gives the equation of the line joining points A(x_1, y_1) and B(x_2, y_2 i.e.,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$$

Example : Find the equation of the line joining the points A (3,1) and B(9,3) using determinants.

Solution : Let P (x,y) be any point on the line joining the points A and B , then the points A,B, and P are collinear

$$\Rightarrow\begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 1 \\ x & y & 1 \end{vmatrix}= 0$$ (operating $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$)

$$\Rightarrow\begin{vmatrix} 3 & 1 & 1 \\ 6 & 2 & 0 \\ x-3 & y-1 & 0 \end{vmatrix}= 0$$

⇒1 [6 (y-1) – 2 (x-3)] =0 expanding the determinant along C_3.

⇒6y – 2x =0

⇒x-3y =0 ,which is the required equation of the line AB.

Example: Consider the points A (3,6), B(6,9) and C(9,12) . Justify whether the points are collinear or not.

Solution : By condition of collinearity, the points will be collinear if

$$\begin{vmatrix} 3 & 6 & 1 \\ 6 & 9 & 1 \\ 9 & 12 & 1 \end{vmatrix} = 0$$

Solving the determinant we get,

$$\Delta = 3^2 \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{vmatrix}$$

Δ = 9[1(3-4) -2(2-3)+1(8-9)]

Δ= 9(-1+2-1) = 9 x 0

Δ = 0

Thus, the points are collinear.

Example: If A (1,3) and B(0,0) and C(k,0) are three points such that area of a triangle ABC is 3 sq.units, then find the value of k.

Solution :

Since Area of triangle ABC=3, so $\frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = 3$

⇒$\frac{1}{2}$[1(0-0)-3(0-k)+1(0-0)]=3

⇒3k=6

Thus, the value of k=2.

Example: If the points A (1,3) and B(0,0) and C(k,0) are collinear, then find the value of k.

Solution :

Since A , B and C are collinear, so by condition of collinearity,

$$\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = 0$$

⇒[1(0-0)-3(0-k)+1(0-0)]=0

⇒3k=0

Thus, the value of k=0.

Example : If A(x_1 ,y_1) , B(x_2 ,y_2), C(x_3 ,y_3) are the vertices of an equilateral triangle with each side equal to a units, then prove that $\begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2 = 3a^4$.

Solution : We know that the area of an equilateral triangle with side 'a' units = $\frac{\sqrt{3}}{4} a^2$ sq. units.

Therefore , the absolute value of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{\sqrt{3}}{4} a^2$ sq. units.

which implies the absolute value of $2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \sqrt{3} a^2$ sq. units.

$\Rightarrow \begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2 = 3a^4$, as required.

Summary:

- The minor M_{ij} of an element a_{ij} in matrix A is the determinant obtained by leaving i^{th} row and j^{th} column of A.
- The cofactor of an element a_{ij} in matrix A is given by $C_{ij} = (-1)^{i+j} M_{ij}$.
- The value of determinant A is obtained by the sum of products of elements of a row (or a column) with corresponding cofactors. For example, $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$.
- If elements of a row (or column) are multiplied with the cofactors of any other row (or column), then their sum is zero. For example, $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$.
- Area of a triangle with vertices (x_1, y_1), (x_2,y_2), (x_3,y_3) is given by the expression

  Area = = $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$
- Three points be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ will be collinear if and only if $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0$
- The equation of the line joining points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by the expression $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$