1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 03 (Class XII, Semester - 1)
Module Name/Title	Determinant - Part 2
Module Id	lemh_10402
Pre-requisites	Basic knowledge about Properties of Determinant and Area of Triangle
Objectives	After going through this lesson, the learners will be able to understand the following: ● Properties of Determinant
Keywords	Properties of Determinant

2. Development Team

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TABLE OF CONTENT:

1. Properties of Determinants:

Property 1: The value of the determinant remains unchanged if its rows and columns are interchanged.

Verification: Now, let A = $\begin{vmatrix} 1 & 2 & 5 \\ 2 & 6 & 7 \\ 3 & 5 & 8 \end{vmatrix}$,

Det A = 1$\begin{vmatrix} 6 & 7 \\ 5 & 8 \end{vmatrix}$ - 2$\begin{vmatrix} 2 & 7 \\ 3 & 8 \end{vmatrix}$ + 5$\begin{vmatrix} 2 & 6 \\ 3 & 5 \end{vmatrix}$ (Expanding along first row)

$= 1(6x8 - 5x7) - 2(2x8 - 3x7) + 5(2x5 - 6x3)$

=1(48- 35) -2(16-21) +5(10-18) = -17

Determinant $|A^T|$ obtained by interchanging rows and columns is

$|A^T|$= $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 6 & 5 \\ 5 & 7 & 8 \end{vmatrix}$

$|A^T|$= 1$\begin{vmatrix} 6 & 5 \\ 7 & 8 \end{vmatrix}$ -2$\begin{vmatrix} 2 & 3 \\ 7 & 8 \end{vmatrix}$+5$\begin{vmatrix} 2 & 3 \\ 6 & 5 \end{vmatrix}$(Expanding along first column)

$|A^T|$=$1(6x8 - 5x7) - 2(2x8 - 3x7) + 5(2x5 - 6x3)$ = -17

$|A|$=$|A^T|$.

Property 2. If any two rows or any two columns of a determinant are interchanged, then the value of the determinant changes from plus to minus and vice versa.

Verification: Let A = $\begin{vmatrix} 1 & 2 & 5 \\ 2 & 6 & 7 \\ 3 & 5 & 8 \end{vmatrix}$, then B= $\begin{vmatrix} 2 & 6 & 7 \\ 1 & 2 & 5 \\ 3 & 5 & 8 \end{vmatrix}$ (Rows 1 and 2 are interchanged).

Then it can be verified that A= - B

Now we know ,

$|A|= 1\begin{vmatrix}6 & 7\\5 & 8\end{vmatrix} -2\begin{vmatrix}2 & 7\\3 & 8\end{vmatrix} +5\begin{vmatrix}2 & 6\\3 & 5\end{vmatrix}$ (Expanding along first row)

$=1(6x8-5x7)-2(2x8-3x7)+5(2x5-6x3)$

=1(48-35) -2(16-21) +5(10-18) = -17

Now, $|B| = 2\begin{vmatrix}2 & 5\\5 & 8\end{vmatrix} - 6\begin{vmatrix}1 & 5\\3 & 8\end{vmatrix} + 7\begin{vmatrix}1 & 2\\3 & 5\end{vmatrix}$

$|B|=2(2x8-5x5)-6(1x8-3x5)+7(1x5-3x2)=17$

Thus, it has been verified that A= -B

Property 3. If any two rows or (columns) of a determinant are identical, then its determinant is zero i.e. $|A|$ = 0

Verification: Let A $=\begin{vmatrix}1 & 2 & 5\\3 & 2 & 6\\1 & 2 & 5\end{vmatrix}$ be a matrix having first and third rows identical.

Then its determinant is $|A| = 1\begin{vmatrix}2 & 6\\2 & 5\end{vmatrix} -2\begin{vmatrix}3 & 6\\1 & 5\end{vmatrix} + 5\begin{vmatrix}3 & 2\\1 & 2\end{vmatrix}$

$|A|$= -2-18+20 = 0

Property 4. If each element of a row (column) of a determinant is multiplied by a constant k, then the value of the new determinant is k times the value of the original determinant.

Verification : Let A $=\begin{vmatrix}3 & 2 & 2\\1 & 4 & 5\\1 & 1 & 1\end{vmatrix}$ be a determinant and B be a determinant obtained from A by multiplying each element of third row by the same constant 2, then B= $\begin{vmatrix}3 & 2 & 2\\1 & 4 & 5\\2 & 2 & 2\end{vmatrix}$

Now A=3(4-5)-2(1-5)+2(1-4) = -1

and B = 3(8-10)-2(2-10)+2(2-8) =-2

B = 2A

Note: Let square matrix A =$[a_{ij}]$ is of order n, then |kA|= k^n |A| as here each element of matrix is multiplied by constant which can be taken common from each row.

Example: Let A = $\begin{vmatrix} 4 & 6 & 8 \\ 4 & 10 & 12 \\ 16 & 48 & 64 \end{vmatrix}$

A = 128

Now, we take 2 common from all the three rows to obtain

A = $2^3\begin{vmatrix} 2 & 3 & 8 \\ 2 & 5 & 6 \\ 8 & 24 & 32 \end{vmatrix}$

A = $8\begin{vmatrix} 2 & 3 & 8 \\ 2 & 5 & 6 \\ 8 & 24 & 32 \end{vmatrix}$

A = 8 x 16 = 128

As the value of determinant $\begin{vmatrix} 2 & 3 & 8 \\ 2 & 5 & 6 \\ 8 & 24 & 32 \end{vmatrix}$= 16

Property 5. Let A be a determinant such that some or all elements of a row (column) of A are expressed as the sum of two or more terms, then the determinant of A can be expressed as the sum of the of two or more determinants.

Verification : Let A = $\begin{vmatrix} 1+a & 2+b & 3+c \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$be a determinant of order 3 such that each element in a first row of A is the sum of two elements, then,

A=$\begin{vmatrix} 1+a & 2+b & 3+c \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Expanding the determinant along first row

A= (1 + a)(3 – 2) – (2 + b)(6 – 1) + (3 + c)(4 – 1)

A = 1 + a – 10 – 5b + 9 + 3c = a – 5b + 3c…………………………………………..(1)

Further, A can also be expressed as sum of two determinants as follows

A = $\begin{vmatrix} 1+a & 2+b & 3+c \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

|A| = 0 + a(3-2) –b(6-1) +c(4-1) = a-5b+3c……………………………..(2)

From (1) and (2) , the property can be verified.

Property 6. (On addition of determinants): If, to each element of any row or column of a determinant, the equimultiples of corresponding elements of other row (or column) are added, then value of determinant remains the same, i.e., the value of determinant remain same if we apply the operation $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$.

Verification:

Let A= $\begin{vmatrix} x & y & z \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$.

And let B= $\begin{vmatrix} x+3a & y+3b & z+3c \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$

Here, we have multiplied the elements of the second row R_2 by a constant 3 and added them to the corresponding elements of the first row R_1.

Symbolically, we write this operation as $R_1 \to R_1 + 3R_2$.

Now, again B= $\begin{vmatrix} x & y & z \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 3a & 3b & 3c \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$,using property 5

$|B| = \begin{vmatrix} x & y & z \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} + 3\begin{vmatrix} a & b & c \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x & y & z \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} + 3 \text{ x } 0 = \begin{vmatrix} x & y & z \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$

Hence, $A = B$

Evaluation of Determinants :

If A is a square matrix of order 2, then its determinant can be easily found. But to evaluate determinants of square matrices of higher orders, we should always try to introduce zeros at maximum number of places in a particular row (column) by using the properties given above and then we should expand the determinant along that row (column).

Example : (Solution of Determinant Equations)

Solve : $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$

Solution : Let $\Delta = \begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix}$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$,we get

$$\Delta = \begin{vmatrix} 3a-x & a-x & a-x \\ 3a-x & a+x & a-x \\ 3a-x & a-x & a+x \end{vmatrix}$$

Taking (3a-x) common from C_1,

$$\Delta = (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 1 & a+x & a-x \\ 1 & a-x & a+x \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and$R_3 \rightarrow R_3 - R_1$

We get

$$\Delta = (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix}$$

On expanding, we get

$$(3a-x)\ x[1 \begin{vmatrix} 2x & 0 \\ 0 & 2x \end{vmatrix}] = 0$$

$4x^2 (3a-x) = 0$

$\Rightarrow x = 0, 3a.$

Example: Without expanding show that

$$\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$$

Solution: Let $\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$. Then taking (-1) common from each row

$$\Delta = (-1)^3 \begin{vmatrix} o & -a & b \\ a & 0 & c \\ -b & -c & o \end{vmatrix}$$

$\Delta = -\begin{vmatrix} o & -a & b \\ a & 0 & c \\ -b & -c & o \end{vmatrix} = -\Delta$ (Using property 1 of determinants)

We get, Δ=-Δ

0 = 2Δ

Δ = 0

Example: Show that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$

Solution: Let determinant $\Delta = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

Multiplying C_1, C_2 and C_3 by a, b, c respectively, we get

$\Delta = \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & bca & cab \end{vmatrix}$

$\Delta = \frac{abc}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$ (Taking abc common from R_3)

Applying $R_2 \leftrightarrow R_3$

$\Delta = -\begin{vmatrix} a^2 & b^2 & c^2 \\ 1 & 1 & 1 \\ a^3 & b^3 & c^3 \end{vmatrix}$ using property 2.

and $R_1 \leftrightarrow R_2$

$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$ using property 2

Example: Show that $\begin{vmatrix} a & b & c \\ a+x & b+y & c+z \\ x & y & z \end{vmatrix}$ $|z|$=0

Solution: Let $\Delta = \begin{vmatrix} a & b & c \\ a+x & b+y & c+z \\ x & y & z \end{vmatrix}$

$|z|$=$\begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix}$+$\begin{vmatrix} a & b & c \\ x & y & z \\ x & y & z \end{vmatrix}$ by using property 5.

Δ = 0 + 0 (Using property 3)

Example: Without expanding, prove that

$\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$=0

Solution: Applying $R_1 \rightarrow R_1 + R_2$ to determinant,

we get

$\begin{vmatrix} x+y+z & y+z+x & z+x+y \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$= 0

On taking x +y +z common from first row ,

we get

=$\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

= 0 (as two rows are equal and hence using property 3).

Summary:

For any square matrix A , its determinant $|A|$ satisfies the following properties:

- The value of the determinant remains unchanged if its rows and columns are interchanged.
- $|A|$ = $|A'|$ where A' is the transpose A.
- If any two rows or any two columns of a determinant are interchanged, then the value of the determinant changes from plus to minus and vice versa.
- If any two rows or (columns) of a determinant are identical, then its determinant is zero .
- If each element of a row (column) of a determinant is multiplied by a constant k, then the value of the new determinant is k times the value of the original determinant.
- A = $[a_{ij}]_{3\times3}$, then $|kA| = k^3\,|A|$.
- Multiplying a determinant by k means multiplying elements of only a row (or a column) by k.
- Let A be a determinant such that some or all elements of a row (column) of A are expressed as the sum of two or more terms, then the determinant of A can be expressed as the sum of the of two or more determinants.
- If, to each element of any row or column of a determinant, the equi-multiples of corresponding elements of other row (or column) are added, then value of determinant remains the same.