1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 03 (Class XII, Semester - 1)
Module Name/Title	Determinant - Part 1
Module Id	lemh_10401
Pre-requisites	Basic knowledge about Properties of Determinant and Area of Triangle
Objectives	After going through this lesson, the learners will be able to understand the following: • Basics of Determinants
Keywords	Determinant

2. Development Team

Role	Name	Affiliation
National MOOC Coordinators (NMC)	Prof. Amarendra P. Behera	CIET, NCERT, New Delhi
Program Coordinator	Dr. Indu Kumar	CIET, NCERT, New Delhi
Course Coordinator/ PI	Prof. Til Prasad Sharma	DESM, NCERT, New Delhi
Subject Coordinator	Anjali Khurana	CIET, NCERT, New Delhi
Subject Matter Expert (SME)	Dr. Monika Sharma	Shiv Nadar University, Noida
Revised by	Manpreet Kaur Bhatia	IINTM College , GGSIP University
Review Team	Prof. Bhim Prakash Sarrah Prof. V.P Singh (Retd.) Prof. SKS Gautam (Retd.)	Assam University, Tezpur DESM, NCERT, New Delhi DESM, NCERT, New Delhi

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1. INTRODUCTION

We know that a system of algebraic equations like

$a_1x + b_1y=c_1$

and $a_2x + b_2y=c_2$

can be represented as $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ in matrix form.

The above system of equations has a unique solution or not is determined by the number a_1b_2-a_2b_1 .

Due to this reason a_1b_2-a_2b_1, written as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, is called **determinant** corresponding to the matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$.

The determinant associated with square matrix A = $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is defined as

$|A|= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} a_1b_2 - a_2b_1$.

The expression on the left is called a determinant and on right i.e., $a_1b_2 - a_2b_1$ is called the value of determinant. It consists of two rows and two columns, and is called determinant of order 2.

REMARK:

To every square matrix A $=[a_{ij}]$of order n, $a_{ij} \in$ R, we can associate a unique real number called **Determinant of matrix A**, denoted by det A or $|A|$or Δ. This may be thought of as a function which associates each square matrix with a unique real number i.e. f : M→R given by f(A) = det A = $|A|$, where M is the set of square matrices and R is the set of real numbers.

Note :

1. A determinant is denoted usually by Δ
2. Only square matrices have a determinant.
3. For matrices A, $|A|$is read as determinant of A not modulus of A.

2. VALUE OF A DETERMINANT :

Determinant of a matrix of order one

Let A =$[a]$ be the square matrix of order 1, then det A =$|a|$ and the value of the determinant is the number itself i.e., $|a|$= a.

Example: Consider a matrix A=$[-3]$. The determinant associated with this matrix is given by Δ =$|-3|$ = -3.

Determinant of a matrix of order two

Let A =$\begin{bmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{bmatrix}$ be a square matrix of order two, then

$$\Delta = \begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix}$$

$$\Delta = a_{11}b_{22} - a_{21}b_{22}.$$

Example: Evaluate the determinant A = $\begin{vmatrix} 2 & 2 \\ 4 & 0 \end{vmatrix}$

Solution: Let A = $\begin{vmatrix} 2 & 2 \\ 4 & 0 \end{vmatrix}$

Then Δ =2 x 0 - 4 x 2

Δ= 0 - 8

Δ= - 8

Determinant of a matrix of order three

Let A=$\begin{bmatrix} a_{11} & b_{12} & c_{13} \\ a_{21} & b_{22} & c_{23} \\ a_{31} & b_{32} & c_{33} \end{bmatrix}$be a square matrix of order three. Then $|A|$ = $\begin{vmatrix} a_{11} & b_{12} & c_{13} \\ a_{21} & b_{22} & c_{23} \\ a_{31} & b_{32} & c_{33} \end{vmatrix}$.

The steps to get the value of det A when expanded along the first row are as follows:

Step 1. Multiply first element a_{11} of R_1 by $(-1)^{1+1}$ that is $(-1)^{\text{sum of suffixes in } a_{11}}$ and with the second order determinant obtained by deleting the elements of first row (R_1) and first column (C_1) of $|A|$ as a_{11} lies in R_1 and C_1.i.e., $(-1)^{1+1}a_{11}\begin{vmatrix} b_{22} & c_{23} \\ b_{32} & c_{33} \end{vmatrix}$

Step 2. Multiply second element b_{12} of R_1 by $(-1)^{1+2}$ that is $(-1)^{\text{sum of suffixes in } b_{12}}$and the second order determinant obtained by deleting the elements of first row (R_1) and second column (C_2) of $|A|$ as b_{12} lies in R_1 and C_2 i.e., $(-1)^{1+2}b_{12}\begin{vmatrix} a_{21} & c_{23} \\ a_{31} & c_{33} \end{vmatrix}$

Step 3. Multiply third element c_{13} of R_1 by $(-1)^{1+3}$ that is $(-1)^{\text{sum of suffixes in } c_{13}}$ and the second order determinant obtained by deleting the elements of first row (R_1) and third column (C_3) of $|A|$ as c_{13} lies in R_1 and C_3 i.e., $(-1)^{1+3}c_{13}\begin{vmatrix} a_{21} & b_{22} \\ a_{31} & b_{32} \end{vmatrix}$

Now the expansion of determinant of A, that is $|A|$ written as sum of all three terms obtained in steps 1, 2 and 3 above is given by

det A=$|A|$ = $(-1)^{1+1}a_{11}\begin{vmatrix} b_{22} & c_{23} \\ b_{32} & c_{33} \end{vmatrix}$ + $(-1)^{1+2}b_{12}\begin{vmatrix} a_{21} & c_{23} \\ a_{31} & c_{33} \end{vmatrix}$ + $(-1)^{1+3}c_{13}\begin{vmatrix} a_{21} & b_{22} \\ a_{31} & b_{32} \end{vmatrix}$

det A=$a_{11}\begin{vmatrix} b_{22} & c_{23} \\ b_{32} & c_{33} \end{vmatrix}$-$b_{12}\begin{vmatrix} a_{21} & c_{23} \\ a_{31} & c_{33} \end{vmatrix}$+$c_{13}\begin{vmatrix} a_{21} & b_{22} \\ a_{31} & b_{32} \end{vmatrix}$

det A=$a_{11}(b_{22}c_{33}-b_{32}c_{23})$ -$b_{12}(a_{21}c_{33}-a_{31}c_{23})$ + ($c_{13}(a_{21}b_{32}-b_{22}a_{31})$ ……………………….(1)

Here, the expression of R.H.S is called the expansion of the determinant obtained by expanding along the first row.

Note:

We have expanded the determinant using the first row. However, it can be expanded along any row or any column using the same procedure.

Let's see expansion of determinant along second row and observe what happens:

Now ,let A =$\begin{bmatrix} a_{11} & b_{12} & c_{13} \\ a_{21} & b_{22} & c_{23} \\ a_{31} & b_{32} & c_{33} \end{bmatrix}$be square matrix of order three. Then the steps to get the value of det A are as follows:

Step 1. Multiply first element a_{21} of R_2 by $(-1)^{2+1}$ that is $(-1)^{\text{sum of suffixes in } a_{21}}$ and with the second order determinant obtained by deleting the elements of second row (R_2) and first column (C_1) of $|A|$ as a_{21} lies in R_2 and C_1.i.e., $(-1)^{2+1}a_{21}\begin{vmatrix} b_{12} & c_{13} \\ b_{32} & c_{33} \end{vmatrix}$

Step 2. Multiply second element b_{22} of R_2 by $(-1)^{2+2}$ that is $(-1)^{\text{sum of suffixes in } b_{22}}$ and the second order determinant obtained by deleting the elements of second row (R_2) and second column (C_2) of $|A|$ as b_{22} lies in R_2 and C_2 i.e., $(-1)^{2+2}b_{22}\begin{vmatrix} a_{11} & c_{13} \\ a_{31} & c_{33} \end{vmatrix}$

Step 3. Multiply third element c_{23} of R_2 by $(-1)^{2+3}$ that is $(-1)^{\text{sum of suffixes in } c_{23}}$ and the second order determinant obtained by deleting the elements of second row (R_2) and third column (C_3) of $|A|$ as c_{23} lies in R_2 and C_3 i.e., $(-1)^{2+3}c_{23}\begin{vmatrix} a_{11} & b_{12} \\ a_{31} & b_{32} \end{vmatrix}$

Now the expansion of determinant of A, that is $|A|$ written as sum of all three terms obtained in steps 1,2 and 3 above is given by

det A $=|A|= (-1)^{2+1}a_{21}\begin{vmatrix} b_{12} & c_{13} \\ b_{32} & c_{33} \end{vmatrix}+(-1)^{2+2}b_{22}\begin{vmatrix} a_{11} & c_{13} \\ a_{31} & c_{33} \end{vmatrix}+(-1)^{2+3}c_{23}\begin{vmatrix} a_{11} & b_{12} \\ a_{31} & b_{32} \end{vmatrix}$

det A $= -a_{21}\begin{vmatrix} b_{12} & c_{13} \\ b_{32} & c_{33} \end{vmatrix}+ b_{22}\begin{vmatrix} a_{11} & c_{13} \\ a_{31} & c_{33} \end{vmatrix} - c_{23}\begin{vmatrix} a_{11} & b_{12} \\ a_{31} & b_{32} \end{vmatrix}$

det A $=-a_{21}(b_{12}c_{33}-b_{32}c_{13}) +b_{22}(a_{11}c_{33}-a_{31}c_{13}) -c_{23}(a_{11}b_{32}-b_{12}a_{31})$……………….(2)

Here, the expression of R.H.S is called the expansion of the determinant obtained by expanding along the second row.

Let's see expansion along first column C_1, and observewhat happens

Again, IfA $=\begin{vmatrix} a_{11} & b_{12} & c_{13} \\ a_{21} & b_{22} & c_{23} \\ a_{31} & b_{32} & c_{33} \end{vmatrix}$

Expanding along C_1, we get

det A$=|A|=(-1)^{1+1}a_{11}\begin{vmatrix} b_{22} & c_{23} \\ b_{32} & c_{33} \end{vmatrix}+(-1)^{2+1}a_{21}\begin{vmatrix} b_{12} & c_{13} \\ b_{32} & c_{33} \end{vmatrix}+(-1)^{3+1}a_{31}\begin{vmatrix} b_{12} & c_{13} \\ b_{22} & c_{23} \end{vmatrix}$

det A$= a_{11}\begin{vmatrix} b_{22} & c_{23} \\ b_{32} & c_{33} \end{vmatrix} - a_{21}\begin{vmatrix} b_{12} & c_{13} \\ b_{32} & c_{33} \end{vmatrix}+ a_{31}\begin{vmatrix} b_{12} & c_{13} \\ b_{22} & c_{23} \end{vmatrix}$

det A$=a_{11}(b_{22}c_{33} - b_{32}c_{23}) - a_{21}(b_{12}c_{33} - b_{32}c_{33}) + a_{31}(b_{12}c_{23} - b_{22}c_{13})$………………..(3)

For Exercise

- Verify that the values of determinant $|A|$in (1), (2),(3) are equal.
- Verify that the values of$|A|$ by expanding along R_3, C_2 and C_3 are equal to the value of$|A|$ obtained in (1),(2),(3).

Example: Evaluate the determinant A= $\begin{vmatrix} 3 & 2 & 1 \\ 6 & 8 & 9 \\ 7 & 4 & 5 \end{vmatrix}$

Solution: Following the procedure explained above, we get the determinant as follows**:**

$|A|$= $(-1)^{1+1}3 \begin{vmatrix} 8 & 9 \\ 4 & 5 \end{vmatrix} + (-1)^{1+2}2 \begin{vmatrix} 6 & 9 \\ 7 & 5 \end{vmatrix} + (-1)^{1+3}1 \begin{vmatrix} 6 & 8 \\ 7 & 4 \end{vmatrix}$

$|A|$= $3 \begin{vmatrix} 8 & 9 \\ 4 & 5 \end{vmatrix} -2 \begin{vmatrix} 6 & 9 \\ 7 & 5 \end{vmatrix} + 1 \begin{vmatrix} 6 & 8 \\ 7 & 4 \end{vmatrix}$

$|A|$= 3(8 x 5 -4 x 9)-2(6 x 5-7 x 9) + 1(6 x 4-7 x 8)

$|A|$= 3(40-36) – 2(30-63) + 1(24-56) = 46

The expression of R.H.S is called the expansion of the determinant by the first row.

Example : Evaluate Δ = $\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & 2 \\ 1 & -1 & 5 \end{vmatrix}$

Solution: Expanding along the first column we get,

$|A|$= $1 \begin{vmatrix} 1 & 2 \\ -1 & 5 \end{vmatrix} -4 \begin{vmatrix} 3 & 2 \\ -1 & 5 \end{vmatrix} + 1 \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}$

$|A|$= 1(1 x 5-(-1) x 2) -4(3 x 5-(-1) x 2) + 1(3 x 2-1 x 2)

$|A|$= 1(5+2)-4(15+2)+1(6-2) = -57

Remark: Since expanding along any row or any column will give the same value of determinant, we prefer to expand along the row or column where two or more elements are zero, so that the expression becomes easy.

Let's consider one more example.

Example: Evaluate the determinant $\begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$

Solution: Note that in third column, two entries are zero. So expanding along third column (C_3), We get

$\Delta = 4\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$

Δ=4(-1-12) -0 +0 =- 52.

Example: If A = $\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A|$= 4$|A|$

Solution: Firstly, we evaluate the determinant of A given by

$|A|$= $\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$ = 1 x 2-4 x 2= 2-8 =-6

So 4$|A|$ = 4(-6) = -24 ……………………………..(1)

Further, we consider 2A =2 $\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$ = $\begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$

Then its determinant is given by $|2A|$ = $\begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix}$ = 2 x 4 – 8 x 4 = 8-32 = -24………..(2)

On comparing (1) and (2) we see that $|2A|$= 4$|A|$

Example: Evaluate $\begin{vmatrix} cosA & sinA \\ -sinA & cosA \end{vmatrix}$

Solution : let Δ = $\begin{vmatrix} cosA & sinA \\ -sinA & cosA \end{vmatrix}$

ThenΔ = cosA x cosA – (-sinA) x sinA

Δ= $\cos^2$ A + $\sin^2$ A = 1

Example: Find the values of x for which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix}$ = $\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

Solution: Since $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix}$ = $\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$**,** we evaluate the determinants on both sides and obtain

3 x 1 – x^2 = 3 x 1 - 2 x 4

3 - x^2 = 3 - 8

x^2 = 8

x = 2 $\sqrt{2}$,-2 $\sqrt{2}$

Summary:

- Determinant of matrix A = $[a]_{1\times 1}$ is given by det A =$|a|$
- Determinant of matrix A = $\begin{bmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{bmatrix}_{2\times 2}$ is given by det A = $\begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix}$ = $a_{11}b_{22} - a_{21}b_{22}$.
- Determinant of matrix A = $\begin{bmatrix} a_{11} & b_{12} & c_{13} \\ a_{21} & b_{22} & c_{23} \\ a_{31} & b_{32} & c_{33} \end{bmatrix}_{3\times 3}$ is given by det A = $\begin{vmatrix} a_{11} & b_{12} & c_{13} \\ a_{21} & b_{22} & c_{23} \\ a_{31} & b_{32} & c_{33} \end{vmatrix}$

  = $a_{11}\begin{vmatrix} b_{22} & c_{23} \\ b_{32} & c_{33} \end{vmatrix}$ - $b_{12}\begin{vmatrix} a_{21} & c_{23} \\ a_{31} & c_{33} \end{vmatrix}$+$c_{13}\begin{vmatrix} a_{21} & b_{22} \\ a_{31} & b_{32} \end{vmatrix}$ when expanded along the first row.