1. Details of Module on Matrices and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 03 (Class XII, Semester - 1)
Module Name/Title	Matrices – Part 3
Module Id	Lemh_10303
Pre-requisites	Knowledge about Matrices, Types of matrices and Operation on matrices.
Objectives	After going through this lesson, the learners will be able to understand the following: 1. Transpose of a Matrix 2. Symmetric and Skew symmetric matrices 3. Elementary Operations of a Matrix 4. Invertible Matrices. 5. Summary
Keywords	Transpose, Symmetric, Skew Symmetric, Elementary operation and Inverse.

2. Development Team

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1. Transpose of a Matrix

If A = $[a_{ij}]$ be an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the *transpose* of A. Transpose of the matrix A is denoted by A' or (A^T).

- **Properties of transpose of matrix**

For any matrices A and B of suitable orders, we have

(i) $(A^T)^T = A$,

(ii) $(kA)^T = k\,A^T$ (where k is any constant)

(iii) $(A + B) = A^T + B^T$

(iv) $(A\,B)^T = B^T A^T$

Remark:

- If the matrix A is of order $m \times n$ then the order of matrix A^T is $n \times m$.
- If A is a diagonal matrix then $(A^T) = A$.
- If matrix A is an upper triangular matrix then its transpose will be a lower triangular matrix and conversely.

Example2: If $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix}$, then $A^T = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$

2. Symmetric and Skew Symmetric Matrices

A square matrix A = $[a_{ij}]$ is said to be ***symmetric*** if $A^T = A$, that is, $[a_{ij}] = [a_{ji}]$ for all possible values of i and j.

Example: $$A = \begin{bmatrix} 2 & 1 & \sqrt{3} \\ 1 & 0 & -6 \\ \sqrt{3} & -6 & 3 \end{bmatrix}$$

$A^T = A$

So, A is a symmetric matrix

A square matrix A = [a_{ij}] is said to be ***skew symmetric*** matrix if $A^T = -A$, that is $a_{ji} = -a_{ij}$ for all possible values of i and j.

Example: $$A = \begin{bmatrix} 0 & 3 & 9 \\ -3 & 0 & 1 \\ -9 & -1 & 0 \end{bmatrix}$$

As $A^T = -A$

So, A is a Skew symmetric matrix

Remark:

- ➢ Symmetric and skew symmetric matrix is always a square matrix.
- ➢ The elements on the main diagonal of a skew symmetric matrix are all zero.
- ➢ A matrix which is both symmetric as well as skew symmetric is a null matrix
- ➢ All positive integral powers of a symmetric matrix are symmetric.
- ➢ Every square matrix can be uniquely expressed as the sum of a symmetric matrix and skew-symmetric matrix.
- ➢ If A be a square matrix, then $A^T + A$, AA^T and A^TA are symmetric matrix
- ➢ If A be a square matrix, then $A - A^T$.
- ➢ If A and B are symmetric matrices, then show that AB is symmetric $\quad iff\ AB = BA$

Example: Express matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ as the sum of a symmetric and a skew- symmetric matrix.

Solution: Here $A^T = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$

We can write $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

Let $P = \frac{1}{2}(A + A^T) = \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 6 & 6 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix}$

Now, $P^T = P$. Thus, P is a symmetric matrix

Also, let $Q = \frac{1}{2}(A - A^T) = \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

Now, $Q^T = -Q$. Thus, Q is a Skew - symmetric matrix

$$P + Q = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$$

Thus, A is represented as the sum of a symmetric and a skew symmetric matrix.

Example: Show that the matrix B^TAB is a symmetric or skew- symmetric according as A is symmetric or skew-symmetric.

Solution: $(B^T AB)^T = (AB)^T (B^T)^T$
$\Rightarrow B^T A^T B \ldots\ldots\ldots\ldots\ldots(i)$

Case I when A is a symmetric matrix. Then $A^T = A$

By (i) we get, $(B^T AB)^T = B^T AB$

Case II when A is a skew symmetric matrix. Then $A^T = -A$

By (i) we get, $(B^T AB)^T = -B^T AB$

3. Elementary Operations of a Matrix

There are six operations (transformations) on a matrix, three of which are due to rows and three due to columns, which are known as *elementary operations* or *transformations*.

i. *The interchange of any two rows or two columns*. Symbolically the interchange of i^{th} and j^{th} rows is denoted by $R_i \leftrightarrow R_j$

ii. *The multiplication of the elements of any row or column by a non zero number*. Symbolically, the multiplication of each element of the i^{th} row by $k \neq 0$ is denoted by $R_i \to kR_i$

iii. *The interchange of any two rows or two columns*. Symbolically the interchange of i^{th} and j^{th} rows is denoted by $R_i \leftrightarrow R_j$

iv. *The multiplication of the elements of any row or column by a non zero number*. Symbolically, the multiplication of each element of the i^{th} row by $k \neq 0$ is denoted by $R_i \to kR_i$

v. *The addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non zero number*. Symbolically, the addition to the elements of i^{th} row, the corresponding elements of j^{th} row multiplied by k is denoted by $R_i \to R_i + kR_j$

Example: Apply $R_1 \leftrightarrow R_2$ and $R_1 \to R_1 + 2R_3$ to $A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 2 & 5 \\ 1 & 5 & -2 \end{bmatrix}$

Solution: In $A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 2 & 5 \\ 1 & 5 & -2 \end{bmatrix}$ first we will apply $R_1 \leftrightarrow R_2$

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & -1 & 1 \\ 1 & 5 & -2 \end{bmatrix}$$

Now, applying $R_1 \to R_1 + 2R_3$, we get

$$\Rightarrow A = \begin{bmatrix} 3 & 12 & 1 \\ 3 & -1 & 1 \\ 1 & 5 & -2 \end{bmatrix}$$

4. Invertible Matrices

If A is a square matrix of order *m*, and if there exists another square matrix B of the same order *m*, such that AB = BA = I, then B is called the *inverse* matrix of A and it is denoted by A^{-1}.

Remark:

Rectangular matrix does not possess inverse matrix, since for products BA and AB to be defined and to be equal, it is necessary that matrices A and B should be square matrices of the same order.

- If B is the inverse of A, then A is also the inverse of B.
- Inverse of a square matrix, if it exists, is unique

Example: let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ be two matrices

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Also, BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, B is the inverse of A, i.e. $B=A^{-1}$ and A is inverse of B, i.e. $B^{-1}=A$.

5. Inverse of a matrix through elementary operations

If A is a matrix such that A^{-1}exists, then to find A^{-1} using elementary row operations, write A= IA and apply a sequence of row operation on A = IA till we get, I = BA. The matrix B will be the inverse of A. Similarly, if we wish to find A^{-1} using column operations, then, write A = AI and apply a sequence of column operations on A = AI till we get, I = AB.

Remark:

- In case, after applying one or more elementary row (column) operations on A = IA (A=AI), if we obtain all zeros in one or more rows of the matrix A on L.H.S., then A^{-1} does not exist.

Example: By using elementary operations, find the inverse of the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Solution:

In order to use elementary row operations we may write A = IA.

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to \frac{1}{2} R_1$

$$\begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 5R_1$

$$\begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{-5}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{-5}{2} & 1 & 0 \\ \frac{5}{2} & -1 & 1 \end{bmatrix} A$$

$R_3 \to 2R_3$

$$\begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{-5}{2} & 1 & 0 \\ 5 & -2 & 2 \end{bmatrix} A$$

$R_2 \to R_2 - \frac{5}{2} R_3$

$R_1 \to R_1 + \frac{1}{2} R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A$$

Thus, $$A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

6. Practical problems

Example: Find the values of x, y, z, if the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfy the equation $A^T A = I$

Solution: we have $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$

It is given that $A^T A = I$

$$\Rightarrow \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow 2x^2 = 1, 6y^2 = 1, 3z^2 = 1$$

$$\Rightarrow x = \pm\frac{1}{\sqrt{2}}, y = \pm\frac{1}{\sqrt{6}}, z = \pm\frac{1}{\sqrt{3}}$$

Example: If $A^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ find $A^T - B^T$

Solution: we have $A^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B^T = \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$

$$A^T - B^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$$

Example: show that positive integral powers of a skew –symmetric matrix are skew symmetric and positive even integral powers of a skew symmetric matrix are symmetric.

Solution: Let A be a skew symmetric matrix. Then, $A^T = -A$

We have $(A^n)^T = (A^T)^n$ for all naturals n

$$\therefore (A^n)^T = (-A)^n$$
$$\Rightarrow (A^n)^T = (-1)^n (A)^n$$
$$\Rightarrow (A^n)^T = \begin{cases} A^n & if \quad n \quad even \\ -A^n & if \quad n \quad odd \end{cases}$$

Hence, A^n is symmetric if n is even and skew symmetric if n is odd.

Example: Obtain the inverse of the following matrix using elementary operations

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution: In order to use elementary row operations we may write A = AI

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = A\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_1 \leftrightarrow C_2$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 1 & 3 & 1 \end{bmatrix} = A\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_3 \to C_3 - 2C_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 1 & 3 & -1 \end{bmatrix} = A\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_3 \to C_3 + C_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 2 \end{bmatrix} = A\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_3 \to \frac{1}{2}C_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} = A\begin{bmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$C_1 \to C_1 - 2C_2$$

.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix} = A \begin{bmatrix} -2 & 1 & \frac{1}{2} \\ 1 & 0 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$C_1 \to C_1 + 5C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = A \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -4 & 0 & -1 \\ \frac{5}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$C_2 \to C_2 - 3C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

Hence, $$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

7. Summary

After completing this module learner will be able to understand the concept of operation on matrices, transpose of a matrix, symmetric matrix and skew symmetric matrix, invertible matrices. They will be able to apply the concept practically.