1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 03 (Class XII, Semester - 1)
Module Name/Title	Matrices – Part 2
Module Id	lemh_10302
Pre-requisites	Knowledge about Matrices, Types of matrices and equality of matrices.
Objectives	After going through this lesson, the learners will be able to understand the following: 1. Operations on Matrices • Addition of Matrices • Multiplication of Matrices by a Scalar • Difference of Matrices • Matrix Multiplication 2. Summary
Keywords	Operations on Matrices, scalar, Matrix Multiplication

2. Development Team

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1. Operation on Matrices

- **Addition of Matrices:** If A = [a_{ij}] and B = [b_{ij}] are two matrices of the same order, say $m \times n$. Then, the sum of the two matrices A and B is *defined* as a matrix C =

[c_{ij}]$_{m \times n}$, where $c_{ij} = a_{ij} + b_{ij}$, for all possible values of i and j.

Remark:

- The sum of two matrices is defined only when they have same order. The resultant matrix is also of same order.
- We may observe that addition of matrices is an example of binary operation on the set of matrices of the same order.
- If A= diag(a_1 a_2 $a_{3\ldots}$ a_n)and B=diag(b_1 b_2 $b_{3\ldots}$ b_n), then A+B= diag($a_1 + b_1$ $a_2 + b_2$ $a_3 + b_3$ $_{\ldots}a_{ij} + b_{ij}$). Where *diag* means diagonal marix.

Example 1: If $A = \begin{bmatrix} 2 & 3 & -1 \\ 9 & 5 & 11 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -1 \\ 95 & 2 & 7 \end{bmatrix}$, find A + B

Solution: Clearly, A and B both are the matrices of same order that is 2×3. Now, to find their sum, add the corresponding elements of A and B.

$$A + B = \begin{bmatrix} 2 & 3 & -1 \\ 9 & 5 & 11 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 95 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 2+1 & 3+0 & -1-1 \\ 9+95 & 5+2 & 11+7 \end{bmatrix} = \begin{bmatrix} 3 & 3 & -2 \\ 104 & 7 & 18 \end{bmatrix}$$

Note that the sum of the matrices $\begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 9 & 15 \\ 26 & 1 & 0 \end{bmatrix}$ cannot be determined as the order of the two matrices is not same.

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$$A+B=\begin{bmatrix}2 & 3 & -1\\ 9 & 5 & 11\end{bmatrix}+\begin{bmatrix}1 & 0 & -1\\ 95 & 2 & 7\end{bmatrix}=\begin{bmatrix}2+1 & 3+0 & -1-1\\ 9+95 & 5+2 & 11+7\end{bmatrix}=\begin{bmatrix}3 & 3 & -2\\ 104 & 7 & 18\end{bmatrix}$$

Note that the sum of the matrices $\begin{bmatrix}1 & 0\\ -2 & 3\end{bmatrix},\begin{bmatrix}3 & 9 & 15\\ 26 & 1 & 0\end{bmatrix}$ cannot be determined as the order of the two matrices is not same.

Properties of matrix addition

The addition of matrices satisfies the following properties:

i. **Commutative Law:** If A = [a_{ij}], B = [b_{ij}] are matrices of the same order, say $m \times n$, then A + B = B + A. Now A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]
= [b_{ij} + a_{ij}] (addition of numbers is commutative)
= ([b_{ij}] + [a_{ij}]) = B + A

ii. **Associative Law:** For any three matrices A = [a_{ij}], B = [b_{ij}], C = [c_{ij}] of the same order, say $m \times n$, (A + B) + C = A + (B + C).
Now (A + B) + C = ([a_{ij}] + [b_{ij}]) + [c_{ij}]
= [a_{ij} + b_{ij}] + [c_{ij}] = [(a_{ij} + b_{ij}) + c_{ij}]
= [a_{ij} + (b_{ij} + c_{ij})]
= [a_{ij}] + [(b_{ij} + c_{ij})] = [a_{ij}] + ([b_{ij}] + [c_{ij}]) = A + (B + C)

iii. **Existence of additive identity:** Let A = [a_{ij}] be an $m \times n$ matrix and O be an $m \times n$ zero matrix, then A + O = O + A = A. In other words, O is the additive identity for matrix addition.

iv. (**The existence of additive inverse:** Let A = $[a_{ij}]_{m \times n}$ be any matrix, then we have another matrix as – A = $[-a_{ij}]_{m \times n}$ such that A + (– A) = (– A) + A= O. So
– A is the additive inverse of A or negative of A.

Example2: If $A = diag(2 \quad 1 \quad -4), B = diag(2 \quad \sqrt{3} \quad e)$, then find the additive inverse of (A+B).

Solution: We know If A= diag(a_1 a_2 $a_{3\dots}$ a_n)and B=diag(b_1 b_2 $b_{3\dots}$ b_n), then A+B= diag($a_1 + b_1$ $a_2 + b_2$ $a_3 + b_3$ $_{\dots}a_{ij} + b_{ij}$).

$$\therefore A+B = diag(2+2 \quad 1+\sqrt{3} \quad -4+e)$$

$$\Rightarrow A+B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1+\sqrt{3} & 0 \\ 0 & 0 & -4+e \end{bmatrix}$$

Hence, additive inverse of A+B is $diag(-4 \quad -1-\sqrt{3} \quad 4-e)$.

(as sum of A+B and its negative is a null matrix)

- **Multiplication of a matrix by a Scalar:** If A = $[a_{ij}]_{m \times n}$ is a matrix and k is a scalar, then kA is another matrix which is obtained by multiplying each element of A by the scalar k. In other words, kA = $k\,[a_{ij}]_{m \times n} = [k\,(a_{ij})]_{m \times n}$, that is, ($i$, j)th element of kA is ka_{ij} for all possible values of i and j.

Properties of multiplication of matrix by a scalar:

If A = $[a_{ij}]$ and B = $[b_{ij}]$ be two matrices of the same order, say $m \times n$, and k and l are scalars, then

i. k(A +B) = k A + kB, (ii) ($k + l$)A = k A + l A
ii. k (A + B) = $k([a_{ij}] + [b_{ij}])$
   $= k\,[a_{ij} + b_{ij}] = [k\,(a_{ij} + b_{ij}] = [(k\,a_{ij}) + (k\,b_{ij})]$
   $= [k\,a_{ij}] + [k\,b_{ij}] = k\,[a_{ij}] + k\,[b_{ij}] = k\text{A} + k\text{B}$
iii. $(k + l)\,\text{A} = (k + l)\,[a_{ij}] = [(k + l)\,a_{ij}] + [k\,a_{ij}] + [l\,a_{ij}] = k\,[a_{ij}] + l\,[a_{ij}] = k\,\text{A} + l\text{A}$

Remark:

➢ The negative of a matrix is denoted by –A. We define –A = (– 1) A.

Example3: If X and Y are 2×2 matrices, then solve the following matrix equation for X and Y. $2X+3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}, 3X+2Y = \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix}$

Solution: We have $2X+3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}, 3X+2Y = \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix}$

$$\therefore 3(2X+3Y)-2(3X+2Y)=3\begin{bmatrix}2 & 3\\4 & 0\end{bmatrix}-2\begin{bmatrix}-2 & 2\\1 & -5\end{bmatrix}$$

$$\Rightarrow 5Y=\begin{bmatrix}6+4 & 9-4\\12-2 & 10\end{bmatrix}$$

$$\Rightarrow Y=\frac{1}{5}\begin{bmatrix}10 & 5\\10 & 10\end{bmatrix}$$

$$\Rightarrow Y=\begin{bmatrix}2 & 1\\2 & 2\end{bmatrix}$$

$$2(2X+3Y)-3(3X+2Y)=2\begin{bmatrix}2 & 3\\4 & 0\end{bmatrix}-3\begin{bmatrix}-2 & 2\\1 & -5\end{bmatrix}$$

$$\Rightarrow -5X=\begin{bmatrix}10 & 0\\5 & 15\end{bmatrix}$$

And $\Rightarrow X=\begin{bmatrix}-2 & 0\\-1 & -3\end{bmatrix}$

Hence, $X=\begin{bmatrix}-2 & 0\\-1 & -3\end{bmatrix}$ and $Y=\begin{bmatrix}2 & 1\\2 & 2\end{bmatrix}$

Example4: If $2\begin{bmatrix}3 & 4\\5 & x\end{bmatrix}+\begin{bmatrix}1 & y\\0 & 1\end{bmatrix}=\begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$, find ($x$-$y$).

Solution: Given that $2\begin{bmatrix}3 & 4\\5 & x\end{bmatrix}+\begin{bmatrix}1 & y\\0 & 1\end{bmatrix}=\begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}6 & 8\\10 & 2x\end{bmatrix}+\begin{bmatrix}1 & y\\0 & 1\end{bmatrix}=\begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}6+1 & 8+y\\10+0 & 2x+1\end{bmatrix}=\begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}7 & 8+y\\10 & 2x+1\end{bmatrix}=\begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$$

$$\Rightarrow 8+y=0, 2x+1=5$$

$$\Rightarrow y=-8, x=2$$

$$\therefore x-y=10$$

- **Difference of Matrices:** If A = [a_{ij}], B = [b_{ij}] are two matrices of the same order, say $m \times n$, then difference A – B is defined as a matrix D = [d_{ij}], where $d_{ij} = a_{ij} - b_{ij}$, for all value of i and j. In other words, D = A – B = A + (–1) B, that is sum of the matrix A and the matrix – B.

Example 4: If $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$, find the matrix A

Solution: We have

$$\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}$$

- **Matrix Multiplication:** The *product* of two matrices A and B is *defined* if the number of columns of A is equal to the number of rows of B. Let A = [a_{ij}] be an $m \times n$ matrix and B = [b_{jk}] be an $n \times p$ matrix. Then the product of the matrices A and B is the matrix C of order $m \times p$. To get the $(i, k)^{th}$ element c_{ik} of the matrix C, we take the i^{th} row of A and k^{th} column of B, multiply them element wise and take the sum of all these products. In other words, if A = $[a_{ij}]_{m \times n}$, B = $[b_{jk}]_{n \times p}$, then the i^{th} row of A is [a_{i1} a_{i2} ... a_{in}] and the k^{th} column of B is $\begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix}$, then $c_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k} + ... + a_{in} b_{nk} = \sum_{j=1}^{n} a_{ij} b_{jk}$. The matrix C = $[c_{ik}]_{m \times p}$ is the product of A and B.

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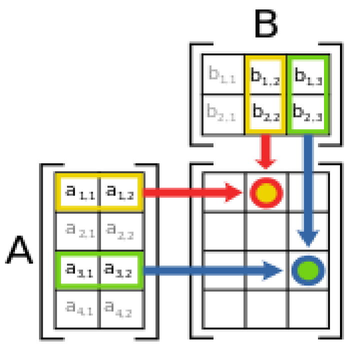

Note that this figure above illustrates diagrammatically the product of two matrices A and B , showing how intersection in the product matrix corresponds to a row of A and a column of B.

Properties of Matrix multiplication

The multiplication of matrices possesses the following properties, which we state without proof.

i. **Associative law:** For any three matrices A, B and C. We have (AB) C = A (BC), whenever both sides of the equality are defined.

ii. **Distributive law:** For three matrices A, B and C.
   - A (B+C) = AB + AC
   - (A+B) C = AC + BC, whenever both sides of equality are defined

iii. **Existence of multiplicative identity:** For every square matrix A, there exist an identity matrix of same order such that IA = AI = A. Now, we shall verify these properties by examples.

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Remark:

- If A and B are two matrices such that AB exist, then BA may or may not exist.
- Matrix multiplication is not commutative .i.e. AB ≠ BA.
- If the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix. Thus, we can get a zero matrix as the product of two non zero matrices.
- Multiplication of diagonal matrices of same order will be commutative

Example 5: If $A=\begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 4 \end{bmatrix}, B=\begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 2 \end{bmatrix}$ then find AB? Is BA defined? If yes, is AB=BA?

Solution: we can observe that the number of rows of A is same as numbers of columns of B, i.e. AB exist. Similarly the numbers of rows of B is same as number of columns of A, i.e. BA also exist.

Now, $AB = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 4 \end{bmatrix} \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 2 \end{bmatrix} = \begin{bmatrix} -2+1-27 & -16-2-6 \\ 1+0+36 & 8+0+8 \end{bmatrix} = \begin{bmatrix} -28 & -24 \\ 37 & 16 \end{bmatrix}$

Similarly, we get $BA = \begin{bmatrix} -2+8 & -1+0 & 3-32 \\ 2+2 & 1+0 & -3-8 \\ 18-2 & 9+0 & -27+8 \end{bmatrix} = \begin{bmatrix} 6 & -1 & -29 \\ 4 & 1 & -11 \\ 16 & 9 & -19 \end{bmatrix}$

Clearly, $AB \neq BA$

Note that the product of the matrices $A = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & 9 & 15 \\ 26 & 1 & 0 \end{bmatrix}$ does not exist as the numbers of rows of A is not equal to the number of columns of B.

Example 6: If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$

and $(A+B)^2 = A^2 + B^2$, then find the values of *a* and *b*.

Solution: Given that $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$

And

$$(A+B)^2 = A^2 + B^2$$

$$\Rightarrow (A+B)(A-B) = A^2 + B^2$$

$$\Rightarrow AA + AB + BA + BB = A^2 + B^2$$

$$\Rightarrow AB + BA = O$$

$$AB = \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix}, BA = \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix}$$

$$now, \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix} + \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a-b+2 & 1-a \\ 2a-2 & 4-b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow 2a-b+2 = 0; 1-a = 0 \Rightarrow a = 1$$

$$\Rightarrow 4-b = 0 \Rightarrow b = 4$$

(as $AB + BA = O$).

Hence, *a=1 ,b=4* satisfy all the four equations

Example7: If matrix $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ and $A^2 = kA$, then write the value of k

Solution:

$$A^2 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} = 4\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 4A$$
$$\Rightarrow 4A = kA \Rightarrow k = 4$$

2. **Practical problems:**

Example 8: In a certain shopping complex there are 3 parking lots. Each parking lot has space to accommodate 103 four wheelers, 50 two-wheelers, 5 bus and 80 bicycles. Using scalar multiplication, find the total vehicles of each kind that can be parked in all parking lots.

Solution: The matrix representing parking space in each parking lot is $\begin{bmatrix} 103 \\ 50 \\ 5 \\ 80 \end{bmatrix}$ The total vehicles that can be parked of each kind in all parking lots are:

$$3\begin{bmatrix} 103 \\ 50 \\ 5 \\ 80 \end{bmatrix} = \begin{bmatrix} 309 \\ 150 \\ 15 \\ 240 \end{bmatrix}.$$

Thus, in 3 parking lots 309 four wheelers, 150 two wheelers, 15 buses and 240 bicycles can be parked.

Example 9: Three schools A, B and C organized a camp for creating awareness on **Plants over Plastic.** They sold handmade fans, mats and plates from recycled material at a cost of rupees 25, rupees 100 and rupees 50 each. The numbers of articles sold are given below:

	SCHOOL A	SCHOOL B	SCHOOL C
HAND FANS	40	25	35
MATS	50	40	50
PLATES	20	30	40

Find the funds collected by each school separately by selling the above articles. Also find the total funds collected for the purpose. Write one value generated by the above situation.

Solution: As we have to find the funds collected by each school. We can write table as:

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 40 & 50 & 20 \\ 25 & 40 & 30 \\ 35 & 50 & 40 \end{bmatrix} \begin{bmatrix} 25 \\ 100 \\ 50 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1000+5000+1000 \\ 625+4000+1500 \\ 875+5000+2000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 7000 \\ 6125 \\ 7875 \end{bmatrix}$$

Funds collected by schools A, B and C are 7000,6125, 7875 rupees respectively.

Total funds collected

= 7000+6125+7875

= 21000 rupees.

The value generated by above is use of eco-friendly articles and a replacement of plastic.

3. **Summary:** After completing this module learners will be able to understand the concept of operation on matrices, sum of two matrices, scalar multiplication, difference of matrices and matrix multiplication. They will be able to apply the concept practically.