1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 03 (Class XII, Semester - 1)
Module Name/Title	Inverse Trigonometric Functions - Introduction ; Properties of Trigonometric Functions - Part 1
Module Id	lemh_10201
Pre-requisites	Functions and types of functions • Composition of functions and inverse of a function • Trigonometric Functions • Properties of Trigonometric functions. • Trigonometric Equations
Objectives	After going through this lesson, the learners will be able to understand the following: • Understand the concept of Domain and Range of Inverse Trigonometric Functions • Define Principal Values of Inverse Trigonometric Functions. • Inverse of sine and cosecant functions
Keywords	Domain, Range, Principal Values, Inverse Trigonometric Functions.

2. Development Team

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1. INVERSE OF A FUNCTION

Corresponding to every bijection (one-one onto function) f: A→B there exist a bijection g: B→A defined by

g(y) =x if and only f(x)=y

The function g: B→A is called the inverse of function f:A→B and is denoted by f^{-1}.

Thus, we have

$f(x)=y \Leftrightarrow f^{-1}(y)=x$

Also, $(f^{-1}of)(x) = f^{-1}[f(x)] = f^{-1}(y) = x$, for all $x \in A$.

And $(fof^{-1})(y) = f[f^{-1}(y)] = f(x) = y$, for all $y \in B$.

2. INVERSE OF TRIGONOMETRIC FUNCTIONS

We know that trigonometric functions are periodic functions, and hence, in general, all trigonometric functions are not bijections. Consequently, their inverses do not exist. However, if we restrict their domains and co-domains, they can be made bijections and we can obtain their inverse. In the following sections, we shall do all these things to obtain the inverses of trigonometric functions.

(i) Inverse of Sine Function

Consider the function f: R → R given by f(x) = sin x. The graph of this function is shown in figure given below.

y=sin(x)

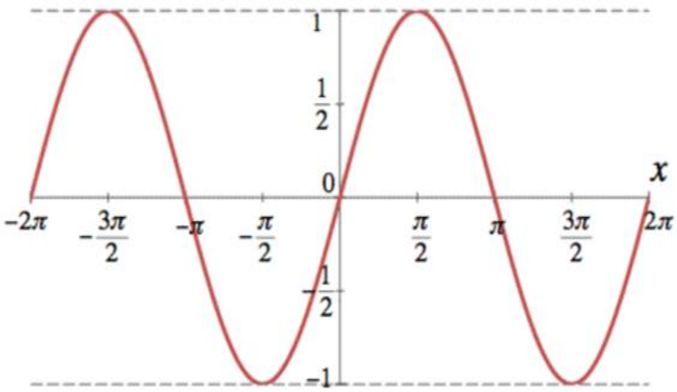

Fig.2.1 (i)

Clearly, it is a many-one into function as it attains same value at infinitely many points and its range [-1,1] is not same as its co-domain. We know that any function can be made an onto function, if we replace its co-domain by its range. Therefore, f:R → [-1,1]: a many-one onto function. In order to make f(x) a one-one function, we will have to restrict its domain in such a way that in that domain the function takes every value between -1 and 1. i.e., if we take the domain as [-π/2, π/2], then f(x) becomes one-one. Thus, f: [-π/2, π/2] → [-1,1] given by f(θ)=sin θ is a bijection and hence invertible.

In the above discussion, we have restricted the domain of sine function to the interval [-π/2, π/2] to make it a bijection. In fact, if we restrict its domain to any one of the intervals [-π/2, π/2], [π/2, 3 π/2], [3π/2, 5 π/2], [-3π/2, - π/2], [-5π/2, -3 π/2] etc., then also it becomes a bijection. We can, therefore, define the inverse of the sine function in each of these intervals. Thus, $\sin^{-1}x$ is a function with domain [-1,1] and range [-π/2, π/2] or [π/2, 3π/2] or [3π/2, 5 π/2] or [-3π/2, - π/2] and so on.

Graph of the sine inverse function and its principal value branch.

Consider the graph of the function $\sin^{-1} : [-1,1] \rightarrow [-\pi/2, \pi/2]$.

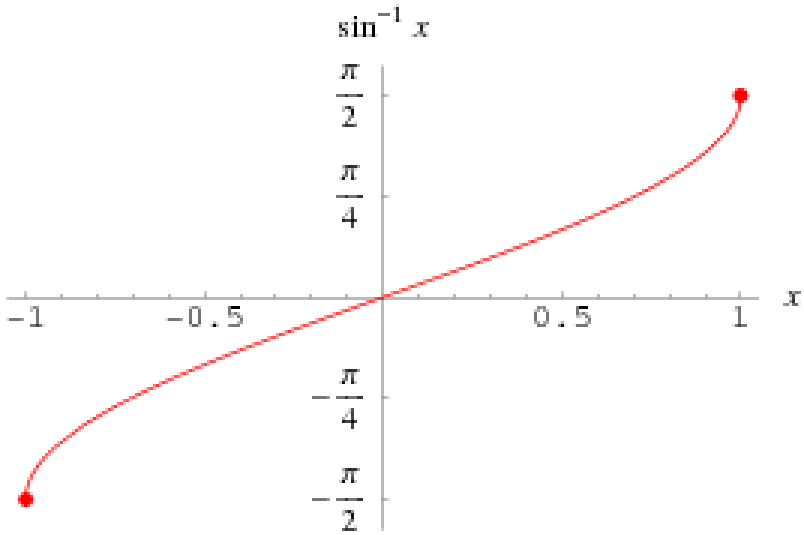

Fig.2.1 (ii)

Corresponding to each such interval, we get a branch of the function $\sin^{-1}x$. The branch of the function $\sin^{-1} : [-1,1] \rightarrow [-\pi/2, \pi/2]$ is called the principal value branch and the value $\sin^{-1} x$ for a given value $x \in [-1,1]$ is called the principal value as shown in above figure.

The graph of $\sin^{-1}$ function can be obtained from the graph of original function by interchanging the coordinate x as the graph of sin function and its inverse, if it exists. They are mirror images of each other in the line mirror y=x ,which is clear from the graph given below.

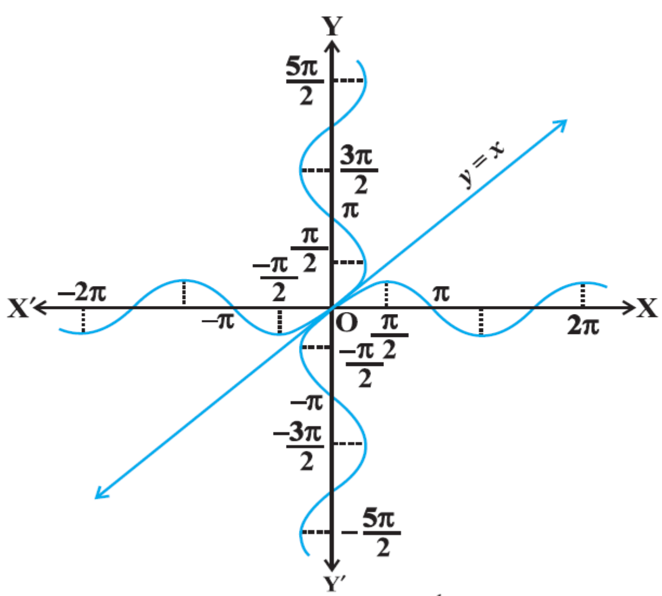

Fig.2.1 (iii)

where, y=sin x,

y=$\sin^{-1}x$,andf(x)=x

NOTE: $\sin^{-1} x$ is not equal to $(\sin x)^{-1}$ or $\frac{1}{Sinx}$.

Example: Find the principal values of $\sin^{-1}(\frac{\sqrt{3}}{2})$

Solution: We know that $\sin^{-1} x$ denotes an angle in the interval $[\frac{-\pi}{2},\frac{\pi}{2}]$ whose sine is x for x $\epsilon[-1,1]$.

Therefore, $\sin^{-1}(\frac{\sqrt{3}}{2})$ = An angle θ $\epsilon[\frac{-\pi}{2}\frac{\pi}{2}]$ such that $\sin\theta=\frac{\sqrt{3}}{2} = \frac{\pi}{3}$.

Example: Find the principal values of $\sin^{-1}(\frac{1}{2})$

Solution: $\sin^{-1}(\frac{1}{2})$ = An angle θ ϵ $[\frac{-\pi}{2}\frac{\pi}{2}]$ such that $\sin\theta = \frac{1}{2} = \frac{\pi}{6}$

Example: Find the domain of the function $f(x) = \sin^{-1}(2x-3)$.

Solution: The domain of $\sin^{-1}x$ is [-1,1].Therefore, $f(x) = \sin^{-1}(2x-3)$ is defined for all x satisfying $-1 \leq 2x-3 \leq 1$

=> $3-1 \leq 2x \leq 3+1$

=> $1 \leq x \leq 2$

=> $x \epsilon [1,2]$

Example: Find the domain of the function $f(x) = \sin^{-1}(-x^2)$.

Solution: The domain of $\sin^{-1}x$ is [-1,1]. Therefore, $f(x) = \sin^{-1}(-x^2)$ is defined for all x satisfying $-1 \leq -x^2 \leq 1$

=> $1 \geq x^2 \geq -1$

=> $0 \leq x^2 \leq 1$=> $x^2 \leq 1$

=> $x^2 - 1 \leq 0$ => $(x-1)(x+1) \leq 0$

=> $-1 \leq x \leq 1$

Hence, the domain of $f(x) = \sin^{-1}(-x^2)$ is [-1,1].

Example: Find the value of $\sin^{-1}[\cos\{\sin^{-1}(-\frac{\sqrt{3}}{2})\}]$

Solution: $\sin^{-1}[\cos\{\sin^{-1}(-\frac{\sqrt{3}}{2})\}]$

$\sin^{-1}[\cos(-\frac{\pi}{3})\}]$ $[\because \sin^{-1}(-\frac{\sqrt{3}}{2} = -\frac{\pi}{3}\}]$

$\sin^{-1}[\cos(\frac{\pi}{3})] = \sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$

(ii) Inverse of Cosecant Function

Consider the graph of cosecant function given below.

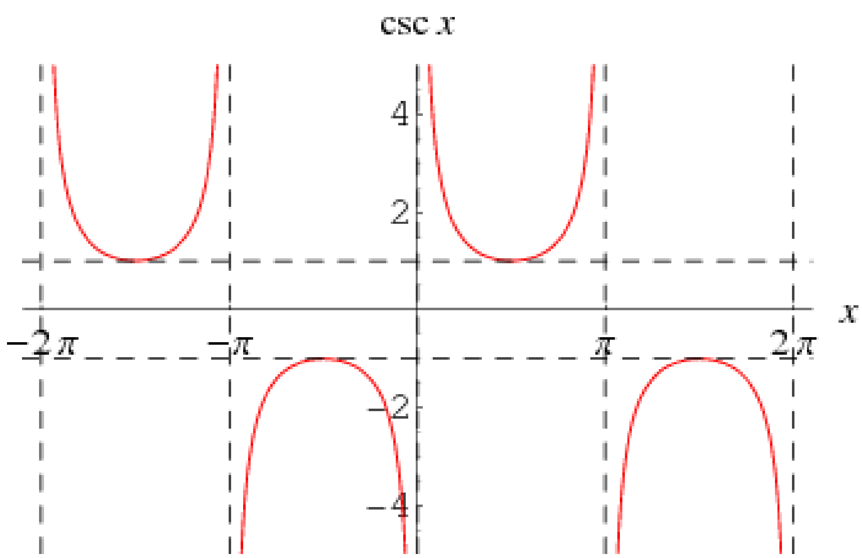

Fig.2.1 (iv)

Since, cosec x= $\frac{1}{sinx}$, the domain of the cosec function is the set {x : x ϵ R and x x≠ nπ ,n=Z} and the range is the set {y : y ϵ R, y ≥ 1 or y ≤ 1}i.e., the set R-(-1, 1). It means that y=cosec x assumes all real values except -1<y<1 and is not defined for integral multiple of π. If we restrict the domain of cosec function to $[-\frac{\pi}{2},\frac{\pi}{2}]$ -{0},then it is one to one and onto with its range as the set R-(-1,1). Actually, cosec function restricted to any of the intervals $[\frac{-3\pi}{2},\frac{-\pi}{2}]$-{- π}, $[\frac{-\pi}{2},\frac{\pi}{2}] - \{0\}, \left[\frac{\pi}{2},\frac{3\pi}{2}\right] - \{\pi\}$ etc., is bijective and its range is the set of all real numbers R-(-1,1).

Consider the graph of y= $\text{cosec}^{-1}(x)$ as shown in figure below.

<u>Graph of $y = cosec^{-1}(x)$:-</u>

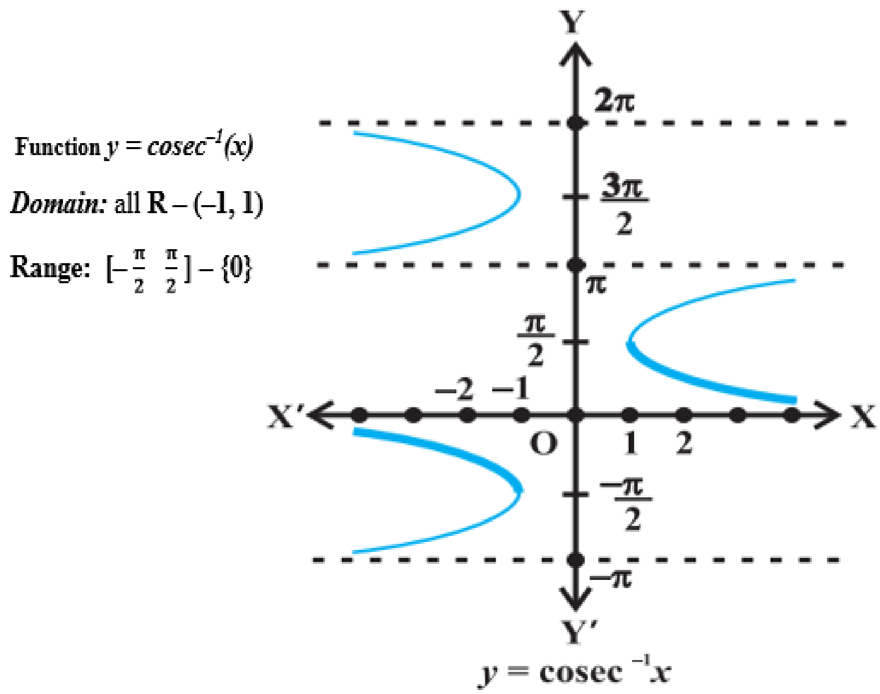

Fig.2.1 (v)

Thus, cosec^{-1} can be defined as a function whose domain is R-(-1,1) and range could be any of the intervals $[\frac{-\pi}{2}, \frac{\pi}{2}] - \{0\}$, $[\frac{\pi}{2}, \frac{3\pi}{2}]$- $\{\pi\}$ etc. The function corresponding to the range $[\frac{-\pi}{2}, \frac{\pi}{2}] - \{0\}$ is called the principal value branch of cosec^{-1}. **We thus have principal branch as shown in graph above of y= cosec^{-1}x by bold blue lines.**

cosec^{-1} : R-(-1,1) → $[\frac{-\pi}{2}, \frac{\pi}{2}] - \{0\}$

Example: Find the principal values of $\text{cosec}^{-1}(2)$.

Solution: For x ϵ (-∞,-1] U [1, ∞), cosec^{-1} x is an angle θ ϵ $[\frac{-\pi}{2}$, 0) U (0,$\frac{\pi}{2}]$ such that cosec θ=x.

∴ $\text{cosec}^{-1}(2)$ = An angle θ ϵ $[\frac{-\pi}{2}$,0) U (0, $\frac{\pi}{2}]$ such that cosec θ=2 = $\frac{\pi}{6}$

Example: Find the set of values of $\text{cosec}^{-1}(-1/2)$.

Solution: We know that cosec^{-1}x is defined for all x ≤ -1 or x ≤ 1.so, $\text{cosec}^{-1}(-1/2)$ is not meaningful. Hence, the set of values of $\text{cosec}^{-1}(-1/2)$ is the null set φ.

3. Summary

- For every bijective (one-one onto) function f: A→B there exist another bijective function g: B→A defined by g(y) =x if and only f(x)=y.The function g is called the inverse of function f and is denoted by f^{-1}.

- The domain and ranges of different trigonometric functions are given below:

Function	**Domain**	**Range (Principal value branches)**
$Y = Sin^{-1} x$	[-1,1]	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$Y = Cosec^{-1} x$	**R – (-1,1)**	$[-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$

- The symbol $sin^{-1}x$ should not be confused with $(\sin x)^{-1}$. In fact, $sin^{-1}x$ is an angle, the value of whose sine is x, similarly for other trigonometric functions.
- The smallest numerical value, either positive or negative, of θ is called the principal value of the function.
- Whenever no branch of an inverse trigonometric function is mentioned, we mean the principal value branch. The value of the inverse trigonometric function which lies in the range of principal branch is its principal value.