## 1. Details of Module and its structure

| Module Detail | |
|---|---|
| Subject Name | Mathematics |
| Course Name | Mathematics 03 (Class XII, Semester - 1) |
| Module Name/Title | Binary Operations - Part 5 |
| Module Id | lemh_10105 |
| Pre-requisites | Mathematical Operations, Commutative & Associative properties of real numbers, Identity & inverse element of real numbers |
| Objectives | After going through this lesson, the learners will be able to understand the following:<br>• Know meaning of binary operations<br>• Understand closed binary operations<br>• Interpret commutative and associative of binary operations<br>• Determine identity and inverse elements of binary operations |
| Keywords | Binary operation, Commutative, Associative, Identity element, Inverse element |

## 2. Development Team

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## 1. **Introduction**

Till now, we have been studying various types of relations and functions which are improper subsets of two non-empty sets. Relations and functions can map one set to another or they may be operated on the same set. We have studied about composition of functions and invertible functions also. All these functions are mappings of set to set and resultant is Cartesian product but is there any type of function which maps Cartesian product of sets to the set? In this module, we will study about this function, which is called binary operation.

## 2. Binary Operations

Let A be a non-empty set. A function * : A x A $\rightarrow$ A is called **binary operation** on set A.

We denote * (a, b) as a * b.

A binary operation * on set A associates each ordered pair (a, b) in A x A to a unique element c in A such that c = a*b

We have many examples of binary operations like addition, subtraction, multiplication, division, HCF, LCM, less than, greater than etc. All these map two elements of a set to a unique single element of a set.

e.g. * : N x N $\rightarrow$ N defined as a*b = a + b

Addition defined on set of natural numbers is a binary operation as sum of any two natural numbers gives a unique natural number.

Let us calculate how many binary operations are possible.

**Number of binary operations on A** = Number of functions from A x A to A

$$= n(A)^{n(A \times A)}$$

$$= n(A)^{n(A) \times n(A)}$$

$$= n(A)^{[n(A)]^2}$$

**Example 1**: Check * : N x N → N defined as a*b = a – b is a binary operation.

**Solution**: Given * : N x N → N defined as a*b = a – b

Let a = 2 and b = 3 then a – b = 2 – 3 = – 1 ∉ N

Thus a * b = a – b on N is not a binary operation.

**Example 2**: Check * : R x R → R defined as a*b = $\frac{a}{b}, b \neq 0$ is a binary operation.

**Solution**: Since $\frac{a}{b} \in R \; \forall a, b \in R \wedge b \neq 0$

Therefore a*b = $\frac{a}{b}$ is a binary operation on R.

**Example 3**: Check * : Z x Z → Z defined as a*b = $a^b$ , b >0 is a binary operation.

**Solution**: Since a*b = $a^b$ ∈ Z , ∀ a , b ∈ Z , b > 0

Therefore, a*b = $a^b$, b > 0 is a binary operation on Z.

**Example 4**: Check * : Z x Z → Z defined as a*b = a – b is a binary operation. If not, then how can it be made binary operation?

**Solution**: Since a*b = a – b ∉ Z for all a, b ∈ $Z$

Let a = 1 , b = 2 then a * b = a – b = 1 – 2 = – 1 ∉ Z

To make it binary operation, we can have following two measures:

**(i)** Replace Z by set of positive integers.

**(ii)** Put a restriction as a > b on Z

**Example 5**: Find the total number of binary operations on set A = {1, 2, 3}

**Solution**: Since n (A) = 3, Number of binary operations = $3^{3^2} = 3^9$

**Example 6**: Let * be a binary operation on set of real numbers 'R' defined as a * b = $a^b + 1$, then evaluate 3 * 2.

**Solution**: 3 ∗ 2 = $3^2 + 1 = 10$

**Example 7**: Let * be a binary operation on set of real numbers 'R' defined as a * b = 3a + 4b, then evaluate (1 * 2)* 5.

**Solution**: (1 * 2)* 5 = (3.1 + 4.2)*5 = 11 * 5 = 3.11 + 4.5 = 33 + 20 = 53

### 3. Commutative Binary Operations

A binary operation * on set A is said to be commutative if a * b = b * a, $\forall \; a, b \in A$

**Example 8**: Check if the binary operation * on R defined as a * b = ab + 5 is commutative.

**Solution**: a * b = ab + 5

b * a = ba + 5 = ab + 5 = a * b

Thus * is commutative binary operation.

**Example 9**: Check if the binary operation * on R defined as a * b = $b^a$ is commutative.

**Solution**: a * b = $b^a$

b * a = $a^b \neq b^a$

Thus a * b ≠ b * a and hence * is not commutative binary operation.

**Example 10**: Check if the binary operation * on R defined as a * b = LCM (a, b) is commutative.

**Solution**: a * b = LCM (a, b)

b * a = LCM ( b, a) = LCM (a, b) = a * b

Thus * is commutative binary operation.

**Example 11**: Check if the binary operation * on R defined as (a, b) * (c, d) = (a + d, b + c) is commutative.

**Solution**: Given (a, b) * (c, d) = (a + d, b + c)

(c, d) * (a, b) = (c + b, d + a)

Since (a, b) * (c, d) ≠ (c, d) * (a, b) and hence * is not commutative binary operation

### 4. Associative Binary Operations

A binary operation * on set A is said to be associative if

$(a * b) * c = a *( b * c), \forall\ a, b, c \in A$

**Example 12**: Check if the binary operation * on R defined as $a * b = ab^2$ is associative.

**Solution:** Given, $a * b = ab^2$

Consider, $(a * b) * c = ab^2 * c = ab^2c^2$

Now, $a *( b * c) = a * (bc^2) = a(bc^2)^2 = ab^2c^4$

Since, $(a * b) * c \neq a *(b * c)$, therefore, * is not associative binary operation.

**Example 13**: Check if the binary operation * on R defined as a * b = HCF (a, b) is associative.

**Solution**: Given a * b = HCF (a, b)

Consider, (a * b) * c = {HCF (a, b)} * c = HCF (a, b, c)

Now, a *( b * c) = a * {HCF (b, c)} = HCF (a, b, c)

Since (a * b) * c = a *( b * c) , therefore, * is associative binary operation.

**Example 14**: Check if the binary operation * on R defined as $a * b = \frac{a-b}{5}$ is associative.

**Solution**: Given $a * b = \frac{a-b}{5}$

Consider, $(a * b) * c = \left(\frac{a-b}{5}\right) * c = \frac{\frac{a-b}{5} - c}{5} = \frac{a-b-5c}{25}$

Now, $a *( b * c) = a * \left(\frac{b-c}{5}\right) = \frac{a - \left(\frac{b-c}{5}\right)}{5} = \frac{5a-b+c}{25}$

Since, $(a * b) * c \neq a *(b * c)$, therefore, * is not associative binary operation.

**Example 15**: Check if the binary operation * on R defined as (a, b) * (c, d) = (a – c, b – d) is associative.

**Solution**: Given (a, b) * (c, d) = (a – c, b – d)

Consider [(a, b) * (c, d)] * (e, f) = (a – c, b – d) * (e, f)

= (a – c – e, b – d – f)

Consider (a, b) * [(c, d) * (e, f)] = (a, b) * (c – e, d – f)

= (a – c – e, b – d – f)

Since, [(a, b) * (c, d)] * (e, f) = (a, b) * [(c, d) * ( e, f)]

Therefore, * is associative binary operation.

## 5. Identity Element of Binary Operation

Let * be a binary operation on set A. If there exists an element e $\in A$ such that

a * e = a = e * a, $\forall\ a, \in A$

then 'e' is called identity element of * on set A.

**Example 16**: Find the identity element of binary operation * on R defined as a * b = $\frac{ab}{7}$

**Solution:** Consider a * b = $\frac{ab}{7}$

Let 'e' be the identity element of * in R then

a * e = a $\Rightarrow \frac{ae}{7} = a$ $\Rightarrow$ e = 7 $\in$ R

e * a = a $\Rightarrow \frac{ea}{7} = a$ $\Rightarrow$ e = 7 $\in$ R

Thus identity element of * is 7 in R

**Example 17**: Find the identity element of binary operation * on R defined as a * b = $a\,b^2$

**Solution:** Consider a * b = $a\,b^2$

Let 'e' be the identity element of * in R then

a * e = a $\Rightarrow$ $ae^2$ = a $\Rightarrow e^2 = 1 \Rightarrow e = \pm 1 \in R$

e * a = a $\Rightarrow$ $ea^2$ = a $\Rightarrow e = \frac{1}{a} \in R$

But $\pm 1 \neq \frac{1}{a}$ , therefore, identity element does not exist for * in R.

**Example 18**: Let A = N x N and * be a binary operation on A defined as (a, b) * (c, d) = (a + c, b + d). Find the identity element of *, if exists.

**Solution**: Let (p, q) $\in A$ be the identity element of * on A, then

(a, b) * (p, q) = (a, b)

$\Rightarrow$ (a + p, b + q) = (a, b)

$\Rightarrow$ a + p = a , b + q = b

$\Rightarrow$ p = 0 = q

Since, (0, 0) ∉ N x N, therefore, identity element does not exist for * in A.

## 6. Inverse Element of Binary Operation

Let * be a binary operation on set A and e be the identity element of * in A. If there exists an element b ∈ $A$ such that

a * b = e = b * a, $\forall\ a, b \in A$

then 'b' is called inverse element of 'a' in set A and 'a' is called invertible element.

**Example 19**: Find the inverse element of binary operation * on R defined as a * b = $\frac{ab}{7}$

**Solution**: e = 7 (refer to example 16 above)

Let 'b' be the inverse of 'a' of * in R, then

a * b = e $\Rightarrow \frac{ab}{7}=7 \quad \Rightarrow b=\frac{49}{a}\in R$

b * a = e $\Rightarrow \frac{ba}{7}=7 \quad \Rightarrow b=\frac{49}{a}\in R$

Thus inverse element of a is $\frac{49}{a}$

**Example 20**: Find the inverse element of binary operation * on R defined as a * b = $ab^2$

**Solution**: Since identity element does not exist (refer example 17 above), therefore, inverse element does not exist.

## 7. Summary

- Let A be a non-empty set. A function * : A x A → A is called binary operation on set A.
- A binary operation * on set A associates each ordered pair (a, b) in A x A to a unique element c in A such that c = a*b
- Number of binary operations on A = $n(A)^{[n(A)]^2}$

- A binary operation * on set A is said to be commutative if a * b = b * a, $\forall\ a, b \in A$
- A binary operation * on set A is said to be associative if (a * b) * c = a *( b * c) $\forall\ a, b, c \in A$
- Let * be a binary operation on set A. If there exists an element e $\in A$ such that a * e = a = e * a $\forall\ a, \in A$ then 'e' is called identity element of * on set A.
- Let * be a binary operation on set A and e be the identity element of * in A. If there exists an element b $\in A$ such that a * b = e = b * a, $\forall\ a, b \in A$ then 'b' is called inverse element of 'a' in set A and 'a' is called invertible element.