1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 03 (Class XII, Semester - 1)
Module Name/Title	Invertible Functions - Part 4
Module Id	lemh_10104
Pre-requisites	Bijections, Composition of functions
Objectives	After going through this lesson, the learners will be able to understand the following: • Apply the concept of bijection to determine inverse of a function. • Apply the concept of composition of functions to determine inverse of a function
Keywords	Invertible function

2. Development Team

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Table of Contents :

1. Introduction

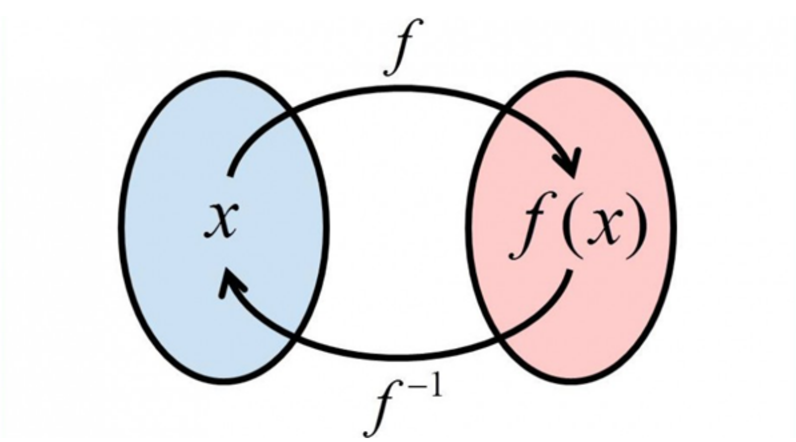
In life there are many situations where we wish to invert the situations. But it is not possible but in mathematics some of the functions can be inverted.

Till now, we have studied about various types of relations and functions, compositions of functions etc. In this module, we will learn how to determine inverse of a function and its properties.

2. Invertible Functions

A function $f : X \to Y$ is said to be invertible if there exists another function $g : Y \to X$ such that $gof = I_X$ and $fog = I_Y$

The function *g* is called inverse of *f* and is denoted by f^{-1}

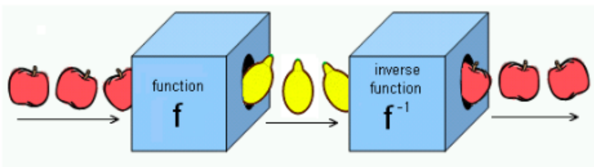


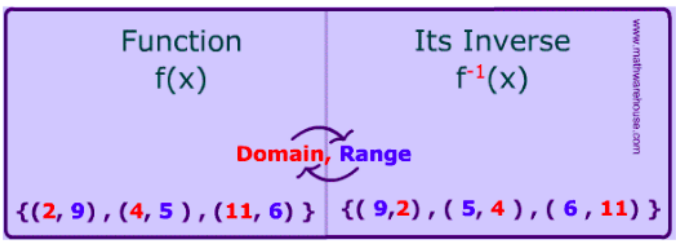
Source: http://blog.brightstorm.com/how-to-find-an-inverse-function/

We can also say that if f is invertible then f must be one-one and onto, i.e., f must be a bijection.

$$y=f(x)\Longleftrightarrow x=f^{-1}(y)$$

Inverse of a function means reversing the process and getting the original thing back.





Source: https://teacher.desmos.com/activitybuilder/custom/57c4877db4f093fa1ae32ef2

Let us learn this concept by solving problems also.

Example 1: Let a bijection *f* is given as *f* = {(1, 4), (2, 9), (3, 16)}. Find f^{-1} .

Solution: Given *f* = {(1, 4), (2, 9), (3, 16)}

Since $y=f(x)\Longleftrightarrow x=f^{-1}(y)$ then *f* (1) = 4 $\Longrightarrow f^{-1}(4)=1$ and so on

Thus f^{-1} = {(4, 1), (9, 2), (16, 3)}

Example 2: Let $f:R\rightarrow R$ be a bijection defined as $(x)=\frac{2x+3}{7}$. Find $f^{-1}(x)$.

Solution: We know y = $f(x)=\frac{x+3}{7}$

$$\Longrightarrow y=\frac{2x+3}{7}$$

$$\Longrightarrow 7y=2x+3$$

$$\Longrightarrow 7y-3=2x$$

$$\Longrightarrow x=\frac{7y-3}{2}$$

Since $y=f(x) \Longleftrightarrow x=f^{-1}(y)=\frac{7y-3}{2}$

Thus $f^{-1}(x)=\frac{7x-3}{2}$

Example 3: Show that the function $f: R \rightarrow R$ defined as $f(x)=\frac{7-3x}{4}$ is invertible and hence find inverse of the function.

Solution: Let $f(a)=f(b)$

$$\Longrightarrow \frac{7-3a}{4}=\frac{7-3b}{4}$$

$$\Longrightarrow 7-3a=7-3b$$

$$\Longrightarrow 3a=3b$$

$$\Longrightarrow a=b$$

Thus *f* is one-one.

Let $y \in R$ be such that y = *f*(x)

$$\Longrightarrow y=\frac{7-3x}{4}$$

$$\Longrightarrow 4y=7-3x$$

$$\Longrightarrow x=\frac{7-4y}{3} \in R$$

Thus f is onto

Hence f is bijective.

Hence f is invertible and $f^{-1}(x)=\frac{7-4x}{3}$

Example 4: Show that the function $\overset{+¿+ \rightarrow R_{\square}}{f: R_{\square}}$ defined as $f(x)=x^2+9$ is invertible and hence find inverse of the function.

Solution: Let $f(a)=f(b)$

$\Longrightarrow a^2+9=b^2+9$

$\Longrightarrow a^2=b^2$

$\Longrightarrow a=b \, as \, a, b \in R_{\square}$

Thus *f* is one-one.

Let $y \in R$ be such that y = *f*(x)

$\Longrightarrow y=x^2+9$

$\Longrightarrow x=\sqrt{y-9} \, as \, x \in R_{\square}$

Thus f is onto

Hence f is bijective.

Hence f is invertible and $f^{-1}(x)=\sqrt{y-9}$

Example 5: Show that the function $\overset{+\to R-\{1\}}{f:R_{\square}}$ defined as $f(x)=\frac{x+2}{x}, x\neq 0$ is invertible and hence find inverse of the function.

Solution: Let $f(a)=f(b)$

$\Longrightarrow \frac{a+2}{a}=\frac{b+2}{b}$

$\Longrightarrow ab+2b=ab+2a$

$\Longrightarrow 2b=2a$

$\Longrightarrow a=b$

Thus *f* is one-one.

Let $+¿ y \in R_{\square}$ be such that y = *f*(x)

$\Longrightarrow y=\frac{x+2}{x}$

$\Longrightarrow xy=x+2$

$\Longrightarrow xy-x=2$

$\Longrightarrow x(y-1)=2$

$\Longrightarrow x=\frac{2}{y-1}, y\neq 1$

Thus f is onto

Hence f is bijective.

Hence f is invertible and $f^{-1}(x)=\frac{2}{y-1}$

Example 6: Show that the function $\begin{matrix} +¿+\to R_{\square} \\ f: R_{\square} \end{matrix}$ defined as $f(x)=\sqrt{4x^2-7}$ is invertible and hence find inverse of the function.

Solution: Let $+¿\, y \in R_{\square}$ be such that y = *f*(x)

$$\Longrightarrow y=\sqrt{4x^2-7}$$

$$\Longrightarrow y^2=4x^2-7$$

$$\Longrightarrow x=\sqrt{\frac{y^2+7}{4}}, x \in R_{\square}$$

$$Let\ g(y)=\sqrt{\frac{y^2+7}{4}}$$

Consider $fog(y)=f(g(y))=f\left(\sqrt{\frac{y^2+7}{4}}\right)$

$$\sqrt{4\left(\frac{y^2+7}{4}\right)-7}$$

$$\sqrt{y^2+7-7}$$

= y

Consider $gof(x)=g(f(x))=g\left(\sqrt{4x^2-7}\right)$

$$\sqrt{\frac{\left(\sqrt{4x^2-7}\right)^2+7}{4}}$$

$$\sqrt{\frac{4x^2}{4}}$$

= x

Thus *gof*(x) = x and *fog*(y) = y and hence $f^{-1}(x)=\sqrt{\frac{x^2+7}{4}}$

Example 7: Let the function $f: R\to R$ defined as $f(x)=\frac{4x+3}{6x-4}, x\neq\frac{2}{3}$. Show that *fof* (x) = x and also find inverse of *f*. [NCERT]

Solution: $fof(x)=f(f(x))=f\left(\frac{4x+3}{6x-4}\right)$

$$\frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}$$

$$\frac{4(4x+3)+3(6x-4)}{6(4x+3)-4(6x-4)}$$

$$\frac{34x}{34}=x$$

Thus *fof* (x) = x

Since *fof* is an identity function, therefore, inverse of *f*(x) is *f*(x) itself.

Example 8: Let the function $f:R\rightarrow R$ defined as $f(x)=10x+7$. Find the function $g:R\rightarrow R$ such that $gof=fog=I_R$ [NCERT]

Solution: If function *g*(x) satisfies $gof=fog=I_R$ then $g(x)=f^{-1}(x)$

Consider $fog(x)=I_R=x$

$$\Longrightarrow f(g(x))=x$$

$$\Longrightarrow 10.\,g(x)+7=x$$

$$\Longrightarrow g(x)=\frac{x-7}{10}$$

This may also be verified that *gof* (x) = x

3. Properties of Inverse of a Function

- Inverse of a function, if exists, is always unique.

Example 9: Show that inverse of a function is unique.

Solution: Let function $f : A \to B$ be a bijection and, if possible, $g : B \to A$ and $h : B \to A$ be two inverses of *f*.

Let g(y) = a and h(y) = b

g(y) = a $\Longrightarrow$ f(a) = y and h(y) = b $\Longrightarrow$ f(b) = y

Thus f(a) = f(b)

$\Longrightarrow$ a = b as f is one-one

$\Longrightarrow g(y) = h(y)$

Hence inverse of a function is unique.

- Inverse of a bijective function is also bijective function.

Example 10: Let function $f : R \to R$ defined as $f(x) = 3x + 4$ be an invertible function. Find its inverse and show that inverse is also a bijection.

Solution: Let y = f(x)

$\Longrightarrow y = 3x + 4$

$\Longrightarrow x = \frac{y-4}{3}$

Then $g(y) = \frac{y-4}{3}$ is the inverse of f(x).

Now, we will show that g(y) is a bijection.

Let g(a) = g(b)

$\Longrightarrow \frac{a-4}{3} = \frac{b-4}{3}$

$\Longrightarrow a - 4 = b - 4$

$\Longrightarrow a = b$

Thus g(y) is one-one.

Let x = g(y)

$\Longrightarrow x = \frac{y-4}{3}$

$\Longrightarrow y = 3x + 4 = f(x)$

Thus g(y) is onto

Thus g(y) is a bijective function.

- Inverse of an inverse function is function itself, i.e., $\left(f^{-1}\right)^{-1}=f$

Example 11: Let f = {(1, 2), (2, 3), (3, 4), (5, 6)} be an invertible function. Show that $\left(f^{-1}\right)^{-1}=f$.

Solution: Given f = {(1, 2), (2, 3), (3, 4), (5, 6)}

f^{-1} = {(2, 1), (3, 2), (4, 3), (6, 5)}

$\left(f^{-1}\right)^{-1}$ = {(1, 2), (2, 3), (3, 4), (5, 6)} = f

- Let $f: X \rightarrow Y$ and $g: Y \rightarrow X$ be two invertible functions then *gof* is also invertible and $(gof)^{-1}=f^{-1} o\, g^{-1}$

Example 12: Let the function $f: R \rightarrow R$ defined as $f(x)=\frac{7-3x}{4}$ is invertible having inverse function $g: R \rightarrow R$ defined as $g(x)=\frac{7-4x}{3}$. If *gof* is invertible then show that $(gof)^{-1}=f^{-1} o\, g^{-1}$

Solution: Consider $(gof)(x)=g(f(x))=g\left(\frac{7-3x}{4}\right)$

$$g\left(\frac{7-3x}{4}\right)$$

$$= \frac{7-4\left(\frac{7-3x}{4}\right)}{3}$$

$$\frac{7-7+3x}{12}$$

$$\frac{x}{4}$$

Given that $f^{-1}(x)=g(x)=\frac{7-4x}{3}$

$$\Longrightarrow g^{-1}(x) = f(x) = \frac{7-3x}{4}$$

$$\left(f^{-1} o\, g^{-1}\right)(x) = f^{-1}\left(g^{-1}(x)\right)$$

$$\Longrightarrow$$

$$f^{-1}\left(\frac{7-3x}{4}\right)$$

$$\frac{7-4\left(\frac{7-3x}{4}\right)}{3}$$

$$\frac{7-7+3x}{12}$$

$$\frac{x}{4}$$

Thus $(gof)^{-1} = f^{-1} o\, g^{-1}$

4. Real Life Applications

- Undo of any process, e.g., typing on a computer is an inverse process.

Source: http://msofficesupport.blogspot.com/2011/04/undo-action-in-microsoft-word.html

- Untying a knot or opening a sealed package.

Source: https://www.123rf.com/photo_89018744_the-girl-is-untying-the-gift.html

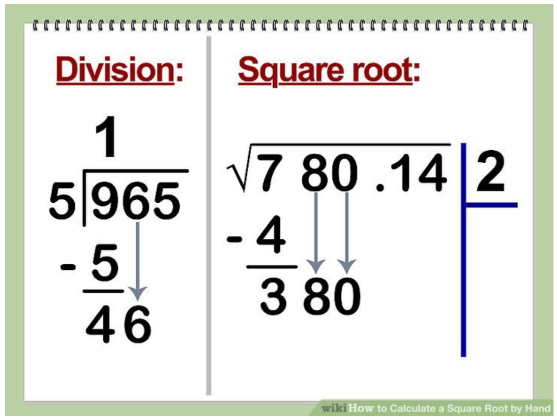
- Solving a puzzle or a problem

Source:https://www.dreamstime.com/stock-images-problem-solving-puzzle-cube-image12468904

- Decoding a code

Source:https://www.codeproject.com/Articles/46174/Computer-Vision-Decoding-a-Morse-Code-Flashing-LED

- Many mathematical functions like square-roots, cube-roots, factorization of equations or constructing an equation using roots, etc are inverse functions.



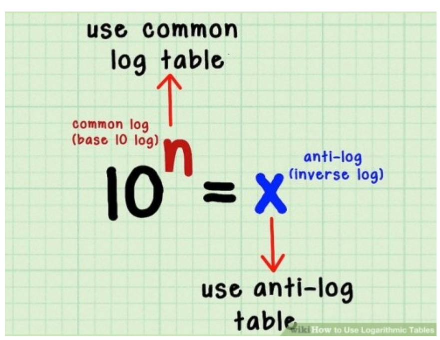
Source: https://www.wikihow.com/Calculate-a-Square-Root-by-Hand

- Inverse trigonometric functions

$$\theta = \cos^{-1}(x) \quad \Leftrightarrow \quad x = \cos(\theta)$$
$$\theta = \sin^{-1}(x) \quad \Leftrightarrow \quad x = \sin(\theta)$$
$$\theta = \tan^{-1}(x) \quad \Leftrightarrow \quad x = \tan(\theta)$$

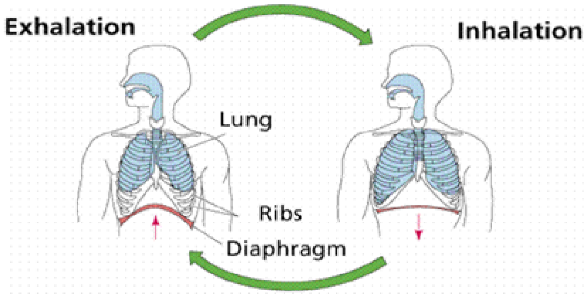
Source: http://tutorial.math.lamar.edu/Extras/AlgebraTrigReview/InverseTrig.aspx

- Taking anti-log



Source: https://www.wikihow.com/Use-Logarithmic-Tables

- Breathing is combined of inhaling and exhaling which are inverse functions of each other.



Source: https://www.differencebetween.com/difference-between-inhalation-and-exhalation/

- Driving in back gear.

Source: https://www.youtube.com/watch?v=ghLaoPdaEME

- Coming back to original posture in a yoga asana.

Source: https://hi.wikipedia.org/wiki/%E0%A4%B8%E0%A5%82%E0%A4%B0%E0%A5%8D%E0%A4%AF_%E0%A4%A8%E0%A4%AE%E0%A4%B8%E0%A5%8D%E0%A4%95%E0%A4%BE%E0%A4%B0

- Getting divorce

Source: https://www.learnvest.com/2013/09/money-divorce-8-dos-and-donts-men-need-to-know

- Coming back home from any other place via same route.

Source: https://culturalawareness.com/repatriation-challenges-faced-coming-home/

- Giving anti-dote medicine

Source: https://www.123rf.com/photo_39929606_diagnosis-antidote-medical-report-with-composition-of-medicaments-red-pills-injections-and-syringe-s.html

5. Summary

- A function $f: X \to Y$ is said to be invertible if there exists another function $g: Y \to X$ such that $gof = I_X$ and $fog = I_Y$
- The function *g* is called inverse of *f* and is denoted by f^{-1}
- We can also say that if f is invertible then f must be one-one and onto, i.e., f must be a bijection.
- $y = f(x) \Longleftrightarrow x = f^{-1}(y)$
- Inverse of a function, if exists, is always unique.
- Inverse of a bijective function is also bijective function.
- Inverse of an inverse function is function itself, i.e., $(f^{-1})^{-1} = f$
- Let $f: X \to Y$ and $g: Y \to X$ be two invertible functions then *gof* is also invertible and $(gof)^{-1} = f^{-1} o\, g^{-1}$