1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 03 (Class XII, Semester - 1)
Module Name/Title	Composition of Functions - Part 3
Module Id	lemh_10103
Pre-requisites	Functions, Domain, Range
Objectives	After going through this lesson, the learners will be able to understand the following: • Determine composition of two or more functions.
Keywords	Composition of functions

2. Development Team

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1. Introduction

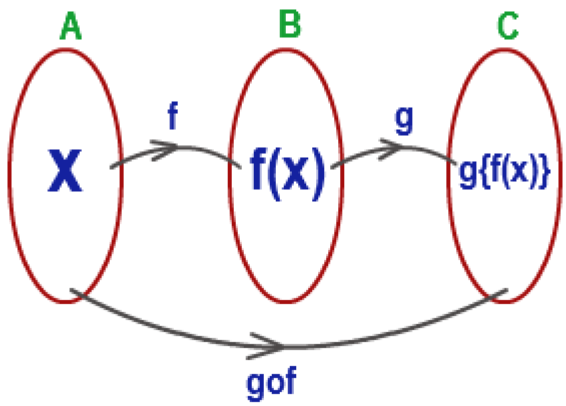
Till now, you were dealing with simple relations and functions, their domains, co-domains, ranges and graphs. You have already studied about types of relations and functions.

Have you ever tried to calculate permutations and combinations problems without using basic mathematical applications of addition, subtraction, multiplication and division? Which mental process goes on while calculating any tougher calculation work? These mental processes use algorithms, which consists of simple mental processes and hence becomes composition of smaller functions.

In this module, we will discuss about functions which are composed of two or functions and their properties.

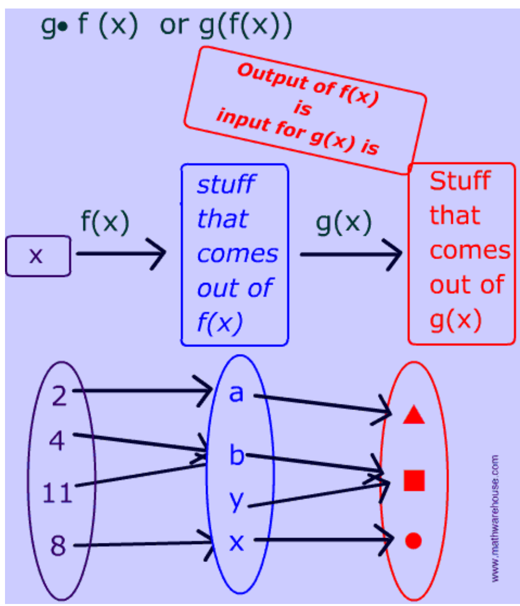
2. Composite Functions

Let A, B and C be three non-empty sets. Let two real functions $f: A \to B$ and $g: B \to C$ be defined. Then the composition of *f* and *g*, denoted as *gof*, is defined as the function $gof: A \to C$ given by $gof(x) = g(f(x)); \forall x \in A$



Source: https://www.tutorvista.com/content/math/composition-of-functions/

Here, range of function *f* becomes domain of function *g*. we can also say that domain of *f* becomes domain of *gof* and range of *g* becomes range of *gof*.



Source: http://www.mathwarehouse.com/algebra/relation/composition-of-function.php

We will do some problems based on this concept to have a better understanding of the concept.

Example 1: Let functions be defined as f = {(1, 2), (2, 3), (3, 4)} and g = {(2, 4), (3, 9), (4, 16)}. Find gof.

Solution: Here, gof(1) = go(f(1))= g(2) = 4

gof(2) = g(f(2)) = g(3) = 9

gof(3) = go(f(3)) = g(4) = 16

Thus gof = {(1, 4), (2, 9), (3, 16)}

Example 2: Let $f, g: R \to R$ be two real functions defined as f(x) = $\sqrt{x}$ and g(x) = $x^2 + 1$. Find gof and fog.

Solution: gof(x) = g(f(x)) = g($\sqrt{x}$) = x + 1

fog(x) = f(g(x)) = f(x^2 + 1) = $\sqrt{x^2+1}$

Example 3: Let $f, g: R \to R$ be two real functions defined as f(x) = sin x and g(x) = $|x|$. Find gof and fog.

Solution: gof(x) = g(f(x)) = g(sin x) = $|\sin x|$

fog (x) = f(g(x)) = f($|x|$) = sin $|x|$

Example 4: Let $f, g: R \to R$ be two real functions defined as $f(x) = |x| + x$ and $g(x) = |x| - x; \forall x \in R$. Then find fog and gof.

[NCERT Exemplar]

Solution: Given $f(x) = |x| + x = \begin{cases} 2x, x \geq 0 \\ 0, x < 0 \end{cases}$

Similarly, $g(x) = |x| - x = \begin{cases} 0, x \geq 0 \\ -2x, x < 0 \end{cases}$

For $x \geq 0$, gof(x) = g(f(x)) = g(2x) = 0

For x< 0, gof(x) = g(f(x)) = g(0) = 0

Thus gof(x) = 0, $\forall x \in R$

For $x \geq 0$, fog(x) = f(g(x)) = f(0) = 0

For x< 0, fog(x) = f(g(x)) = f(–2x) = – 4x

Thus fog(x) = $\begin{cases} 0, x \geq 0 \\ -4x, x < 0 \end{cases}$

Example 5: If $f: R \to R$ be defined as f(x) = $x^2 - 3x + 2$ then find f(f(x)).

[NCERT Exemplar]

Solution: f(x) = $x^2 - 3x + 2$

f(f(x)) = f($x^2 - 3x + 2$)

= $(x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$

= $x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 - 3x^2 + 9x - 6 + 2$

= $x^4 - 6x^3 + 10x^2 - 3x$

Example 6: Let $f: [0,1] \to [0,1]$ be defined as $f(x) = \begin{cases} x, x \text{ is rational} \\ 1 - x, x \text{ is irrational} \end{cases}$ then find fof(x).

[NCERT Exemplar]

Solution: Given $f(x) = \begin{cases} x, x \text{ is rational} \\ 1 - x, x \text{ is irrational} \end{cases}$

Case I: Let x be rational

f(f(x)) = f(f(x)) = f(x) = x

Case II: Let x be irrational

f(f(x)) = f(f(x)) = f(1 – x) = 1 – (1 – x) = x

Thus fof(x) = x $\forall x \in [0,1]$

Example 7: Let $f: R \to R$ be the signum function defined as $f(x) = \begin{cases} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases}$ and $g: R \to R$ be the greatest integer function defined as g(x) = [x]; where [x] is the greatest integer less than or equal to x. Then fog and gof coincide in (0, 1]?

[NCERT Exemplar]

Solution: **Case I**: Let x = 1 then

fog(x) = f(g(x)) = f(1) = 1

gof(x) = g(f(x)) = g(1) = 1

Case II: Let x $\in$ (0, 1)

fog(x) = f(g(x)) = f(0) = 0

gof(x) = g(f(x)) = g(1) = 1

Thus fog and gof coincide only at x = 1

3. Properties of Composite Functions

- Composition of functions is not commutative, in general.

Let real functions $f : A \to B$ and $g : B \to C$ be defined then

$gof \neq fog$

Example 8: Let f(x) = tan x and g(x) = x^2 then show that $gof \neq fog$

Solution: fog(x) = f(g(x)) = f(x^2) = tan(x^2)

gof(x) = g(f(x)) = g(tan x) = $(\tan x)^2$

Thus $gof \neq fog$

If *gof* = *fog* then they become inverse functions of each other. (Invertible functions will be studied in next module).

- Composition of functions is associative.

Let real functions $: A \to B$, $g : B \to C$ and $h : C \to D$ be defined then

$ho(gof) = (hog)of$

Example 9: Let three real functions f, g and h be defined as f(x) = 3x; g(x) = 2x + 4 and h(x) = tan z. Then show that $ho(gof) = (hog)of$

Solution: Consider ho(gof)(x) = ho(g(f(x))

= ho (g(3x))

= ho{2(3x) + 4}

= ho(6x + 4)

= tan (6x +4)

Now, consider ((hog)of)(x) = (hog)(f(x))

$= (hog)(3x)$

$= ho(g(3x))$

$= ho(2(3x) + 4)$

$= ho(6x + 4)$

$= \tan(6x + 4)$

Hence $ho(gof) = (hog)of$

- If real functions $f : A \to B$ and $g : B \to C$ are one-one then $gof : A \to C$ is also one-one.
- If real functions $f : A \to B$ and $g : B \to C$ are onto then $gof : A \to C$ is also onto.
- Given real functions $f : A \to B$ and $g : B \to C$ such that $gof : A \to C$ is one-one then *f* is one-one.
- Given real functions $f : A \to B$ and $g : B \to C$ such that $gof : A \to C$ is onto then is onto.

Example 10: Give examples of two functions $f : N \to Z$ and $g : Z \to Z$ such that *gof* is injective but *g* is not injective.

[NCERT Exemplar]

Solution: Let f(x) = x and g(x) = $|x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$

Clearly, g(x) is not injective as $|1| = |-1| = 1$

Let $x \geq 0$, gof(x) = g(f(x)) = g(x) = x

Let x < 0, gof(x) = g(f(x)) = g(x) = – x

Thus *gof* is injective but *g* is not injective.

Example 11: Give examples of two functions $f : N \to N$ and $g : N \to N$ such that *gof* is surjective but *f* is not surjective.

[NCERT Exemplar]

Solution: Let f(x) = x + 1 and g(x) = $\begin{cases} x - 1, x > 1 \\ 1, x = 1 \end{cases}$

Case I: Let x = 1; gof(x) = g(f(x)) = g(x + 1) = 1

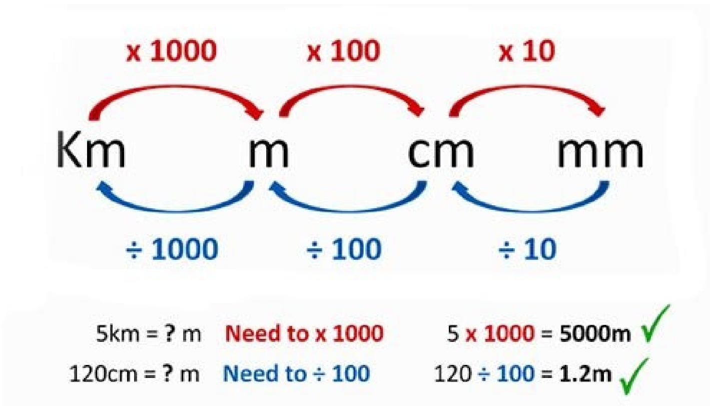
Case II: Let x > 1; gof(x) = g(f(x)) = g(x + 1) = (x + 1) – 1 = x

Being identity function *gof* is surjective.

Range of *f* = N – {1} $\neq$ co-domain and thus *f* is not surjective.

4. Real Life Applications

- Unit conversion in day to day life is an example of composite function.



Source: https://www.proprofs.com/quiz-school/story.php?title=si-unit-conversion

- Buying any product composits its weight and then conversion into its price.

Source: https://www.redrooster.it/en/quality-products/current-topics/

Source: https://www.kwsanantonio.com/news/dont-let-christmas-ruin-your-credit-9-spending-mistakes-that-could-cost-you/

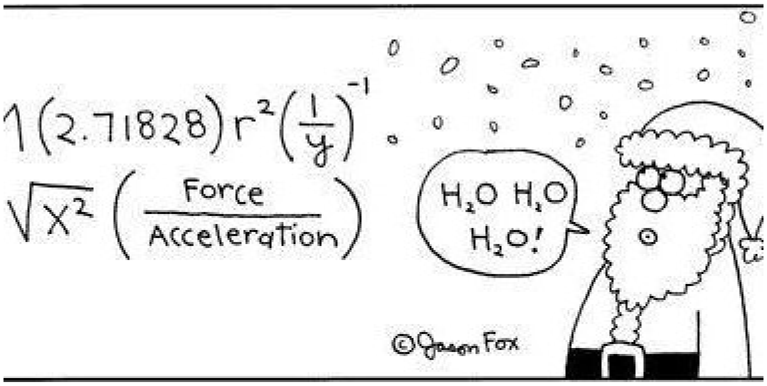
- Packaging unit of any factory. E.g. if we wish to pack any soft drink, then machine will take bottle, fill it with the beverage, cap the bottle, seal it etc.

Source: https://www.youtube.com/watch?v=xLzEdtwnCFk

- Sale given to the customers uses composition of function by seller as well as by the buyer. Specially, the successive discounts offered by the seller. E.g. 50% discount followed by 10% additional for credit card holders.

Source: https://www.nopcommerce.com/blog/78-quick-tip-how-to-get-best-of-the-discounts-functionality-in-nopcommerce-store.aspx

- Any calculation work uses basic calculation followed by application of higher concepts.



Source: https://chemstuff.co.uk/tag/mole-calculations/

- Any machine is made up of many small parts which perform their roles to make the final outcome as desired.

Source: https://www.dreamstime.com/stock-photo-man-operating-cnc-drilling-boring-machine-industry-workshop-industrial-concept-image54296800

- Any process or work done by you or me is composition of many smaller functions. E.g. thinking, writing, walking, talking, singing, running, playing sports, eating, sleeping. Any living organism is performing their day to day simple to complex works by compositing more than two functions.

Source: https://www.shutterstock.com/image-vector/fishing-activity-man-catching-fish-on-722180158

5. **Summary**

- Let A, B and C be three non-empty sets. Let two real functions $f : A \to B$ and $g : B \to C$ be defined. Then the composition of *f* and *g*, denoted as *gof*, is defined as the function $gof : A \to C$ given by $gof(x) = g(f(x)); \forall x \in A$
- Composition of functions is not commutative, in general.
- Let real functions $f : A \to B$ and $g : B \to C$ be defined then $gof \neq fog$
- If *gof* = *fog* then they become inverse functions of each other.
- Composition of functions is associative.
- Let real functions $f : A \to B$, $g : B \to C$ and $h : C \to D$ be defined then
- $ho(gof) = (hog)of$

- If real functions $f: A \rightarrow B$ and $g: B \rightarrow C$ are one-one then $gof: A \rightarrow C$ is also one-one.
- If real functions $f: A \rightarrow B$ and $g: B \rightarrow C$ are onto then $gof: A \rightarrow C$ is also onto.
- Given real functions $f: A \rightarrow B$ and $g: B \rightarrow C$ such that $gof: A \rightarrow C$ is one-one then f is one-one.
- Given real functions $f: A \rightarrow B$ and $g: B \rightarrow C$ such that $gof: A \rightarrow C$ is onto then is onto.