1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 03 (Class XII, Semester - 1)
Module Name/Title	Bijective Functions - Part 2
Module Id	lemh_10102
Pre-requisites	Relations, Functions, Domain, Co-domain, Range
Objectives	After going through this lesson, the learners will be able to understand the following: • Understand injective functions. • Understand surjective functions. • Understand bijective functions.
Keywords	Functions, range, Co-domain, Injective, Surjective, Bijective

2. Development Team

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1. Introduction

In class XI, you have already studied about functions as special type of relations, its domain, co-domain and range. You have already studied about some standard functions and their curve-sketching. In this module, we will study about the types of functions. Before going further, let us recall what is a function?

2. Functions

Let A and B be two non-empty sets. A relation R from A to B is said to be a **function** if for each a $\in$ A, there exists a unique y $\in$ B such that y = f(x).

Thus a function is a relation from a non-empty set A to another non-empty set B if each element a $\in$ A appears in some ordered pair under 'f' and no 'a' is repeated.

Total number of functions from A to B = m^n , where n(A) = m and n(B) = n

A function $f : A \to B$ is called a **real valued function** if B is a subset of R, set of real numbers, and if both A and B are subsets of R then f is a **real function**.

Usually we consider real functions under our study in this module.

We denote function 'f' as $f : A \to B$, where A, B $\subset$ R and y = f(x) for all x $\in$ A and y $\in$ B

Here, y is called **dependent variable** and x is called **independent variable**.

So, set of all values of 'x' is called **domain** and set of all values of 'y' is called **range** of R. Also, 'y' is called image of 'x' and 'x' is called pre-image of 'y'.

i.e., Domain of R = {x} $\subseteq$ A

Range of R = {y} $\subseteq$ B

Co-domain of R = B

Clearly, Range $\subseteq$ Co-Domain

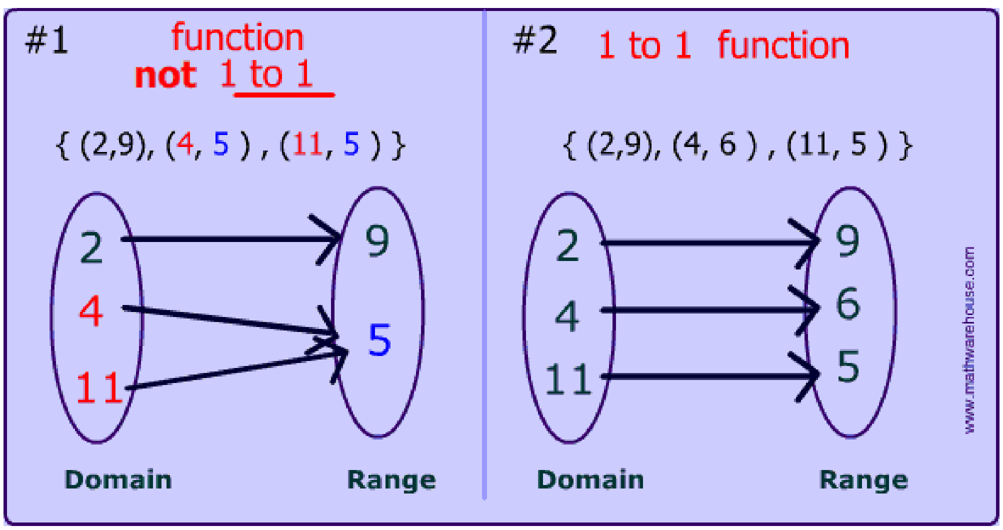
3. Injective Functions

A function $f : A \rightarrow B$ is said to be **injective or one-one function** if images of distinct elements of A under f are distinct, i.e., each element of A has unique image in B, i.e, no two 'a' have same image.

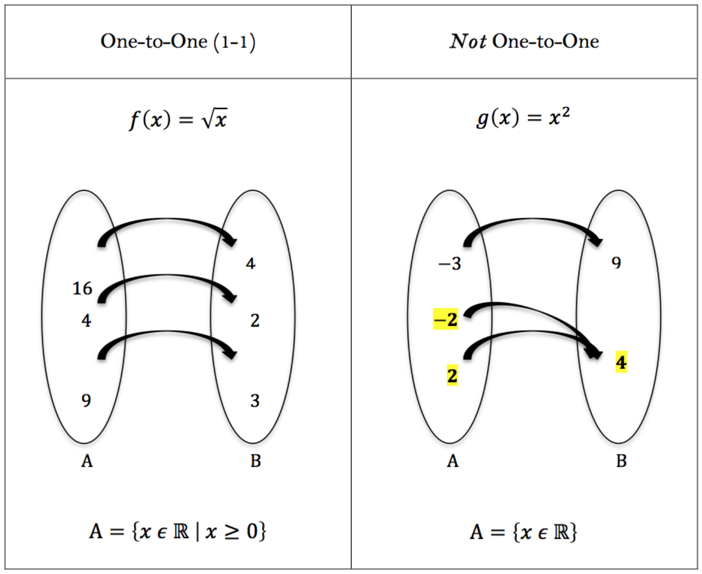
$f : A \rightarrow B$ is one-one $\Leftrightarrow$ f(a) = f(b) $\Rightarrow$ a = b for all a, b $\in$ A

If f is not one-one function then it is called **many-one function**.

Let us understand this by using an arrow diagram.



Source: http://www.mathwarehouse.com/algebra/relation/one-to-one-function.php



Source:http://www.math.toronto.edu/preparing-for-calculus/4_functions/we_3_one_to_one.html

Now, we will strengthen this concept with the help of some examples.

Example 1: Show that the function $f: R \rightarrow R$ is injective where f(x) = 4x – 1 $\forall x \in R$

[NCERT Exemplar]

Solution: Let $f(x_1) = f(x_2)$

$\Rightarrow \quad 4x_1 - 1 = 4x_2 - 1$

$\Rightarrow \quad 4x_1 = 4x_2$

$\Rightarrow \quad x_1 = x_2$

Thus f(x) is injective.

Example 2: Check if the function $f: R \rightarrow R$ given by f(x) = sin x is injective.

Solution: Since f(x) = sin x is periodic function

i.e., $\sin\frac{\pi}{3} = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

i.e., two values of x have same image under ‘f’

Thus f(x) = sin x is not injective.

4. Surjective Functions

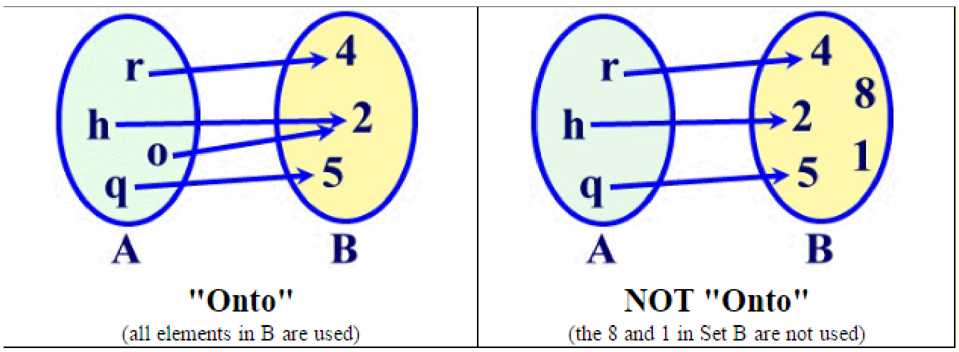
A function $f : A \rightarrow B$ is said to be **onto or surjective** if each element y $\in$ B is image of some element of A under f.

$f : A \rightarrow B$ is said to be onto or surjective $\Leftrightarrow$ for each y $\in$ B, there exists x $\in$ A such that y = f(x)

A function f(x) is onto if **range = co-domain**

If a function is not onto, it is called **into function**.

Let us understand this by using an arrow diagram.



Source: https://sites.google.com/site/aplustoppernotes/home/onto-function-1

Now, we will strengthen this concept with the help of some examples.

Example 3: Show that the function $f : R \rightarrow R$ is sujective where f(x) = 4x – 1 $\forall x \in R$

Solution: Since Range of f(x) = R = Co-domain

Therefore, f(x) is surjective.

Example 4: Check if the function $f : R \rightarrow R$ given by f(x) = sin x is surjective.

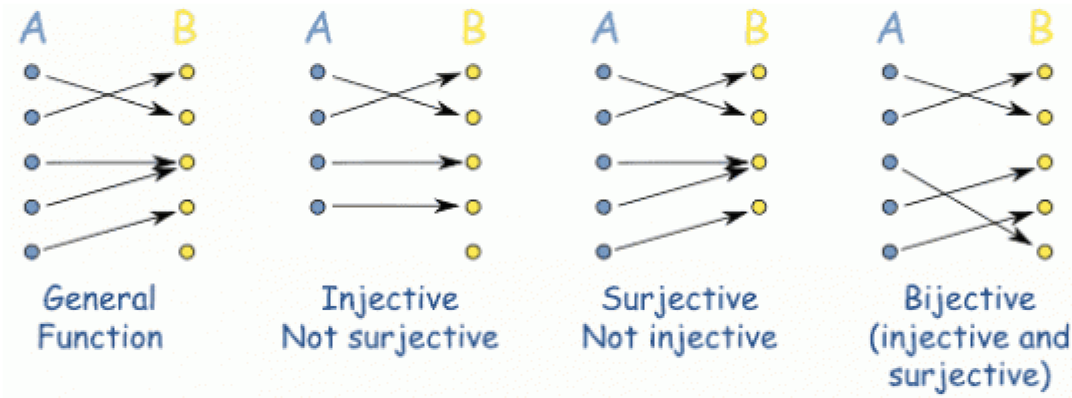
Solution: Since range of f(x) = [– 1, 1] $\neq$ co-domain of f(x)

Thus f(x) is not surjective.

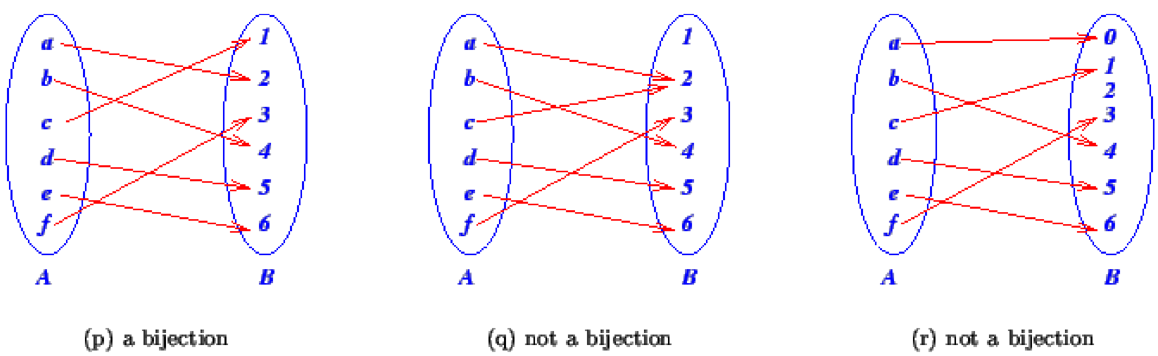
5. Bijective Functions

A function $f : A \to B$ is said to be bijective if f is one-one and onto function.

Let us understand this by using an arrow diagram.



Source: https://notyourmomsfom.wordpress.com/2013/03/04/proving-if-f-is-invertible-then-it-is-bijective/



Source: http://ndp.jct.ac.il/tutorials/discrete/node41.html

Now, we will strengthen this concept with the help of some examples.

Example 5: Let A = [– 1, 1] then check the function defined on set A is one-one, onto or bijective, where f(x) = $|x|$

Solution: Since $|1| = |-1| = 1$

i.e., two values of x have same image under ‘f’

Thus f(x) = $|x|$ is not injective.

Since Range of f(x) = [0, 1] $\neq$ co-domain of f(x)

Thus f(x) is not surjective.

Example 6: Let A = R – {3}, B = R – {1}. Let $f: A \to B$ defined as $f(x) = \frac{x-2}{x-3}, \forall x \in A$.

Then show that f is bijective. [NCERT]

Solution: Let $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

$$\Rightarrow (x_1 - 2)(x_2 - 3) = (x_2 - 2)(x_1 - 3)$$

$$\Rightarrow x_1 x_2 - 3x_1 - 2x_2 + 6 = x_1 x_2 - 2x_1 - 3x_2 + 6$$

$$\Rightarrow -3x_1 - 2x_2 = -2x_1 - 3x_2$$

$$\Rightarrow x_1 = x_2$$

Thus f(x) is one-one.

Let y = f(x)

$$\Rightarrow y = \frac{x-2}{x-3}$$

$$\Rightarrow y(x-3) = x-2$$

$$\Rightarrow xy - 3y = x - 2$$

$$\Rightarrow xy - x = 3y - 2$$

$$\Rightarrow x(y-1) = 3y - 2$$

$$\Rightarrow x = \frac{3y-2}{y-1}$$

Here, y $\neq$ 1, so y $\in$ B

Also, let x = 3 $\Rightarrow 3 = \frac{3y-2}{y-1}$

$$\Rightarrow 3(y-1) = 3y - 2$$

$$\Rightarrow 3y - 3 = 3y - 2$$

$\Rightarrow 3 = 2$, which is absurd

Thus $x=\frac{3y-2}{y-1}\in A=R-\{3\}$

Hence f(x) is surjective.

Hence f(x) is bijective.

Example 7: Let $f:R\rightarrow R$ defined as $f(x)=1+x^2$. Check if f(x) is one-one, onto or bijective. [NCERT]

Solution: Let $f(x_1)=f(x_2)$

$\Rightarrow 1+x_1^2=1+x_2^2$

$\Rightarrow x_1^2=x_2^2$

$\Rightarrow x_1=\pm x_2\, as\, x_1, x_2\in R$

Thus f(x) is not one-one.

Also, Range of f(x) = [1, ∞≠ R

Thus f(x) is not onto.

Hence f(x) is not bijective.

Example 8: Let A and B be two sets. Show that $f:A\times B\rightarrow B\times A$ such that f(a, b) = (b, a) is bijective function. [NCERT]

Solution: Let $f(a_1,b_1)=f(a_2,b_2)$

$\Rightarrow (b_1,a_1)=(b_2,a_2)$

$\Rightarrow b_1=b_2, a_1=a_2$

Thus f is injective.

Let (b, a) ∈ B x A then b ∈ B and a ∈ A ⇒ (a, b) ∈ A x B

Thus, for all (b, a) ∈ B x A, there exists (a, b) ∈ A x B such that f(a, b) = (b, a)

Thus f(a, b) is surjective.

Hence f is bijective map.

Example 9: Let $f : N \to N$ defined as $f(n) = \begin{cases} \frac{n+1}{2}, if\ n\ is\ odd \\ \frac{n}{2}, if\ n\ is\ even \end{cases}; \forall n \in N$. State whether the function is bijective. Justify your answer. [NCERT]

Solution: Here, f(1) = $\frac{1+1}{2}$ = 1 and f(2) = $\frac{2}{2}$ = 1

Thus two distinct values of n have same image under f, therefore, f is not one-one and hence not bijective.

Example 10: Show that the function $f : R \to R$ defined as $f(x) = \frac{x}{1+x^2}, \forall x \in R$ is neither one-one nor onto. [NCERT Exemplar]

Solution: Let $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1}{1+x_1^2} = \frac{x_2}{1+x_2^2}$$

$$\Rightarrow x_1(1+x_2^2) = x_2(1+x_1^2)$$

$$\Rightarrow x_1 + x_1 x_2^2 = x_2 + x_2 x_1^2$$

$$\Rightarrow x_1 - x_2 + x_1 x_2^2 - x_2 x_1^2 = 0$$

$$\Rightarrow (x_1 - x_2) - x_1 x_2 (x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)(1 - x_1 x_2) = 0$$

$$\Rightarrow (x_1 - x_2) = 0 \lor (1 - x_1 x_2) = 0$$

$$\Rightarrow x_1 = x_2 \lor 1 = x_1 x_2$$

Thus f(x) is not injective as $1 = x_1 x_2$ does not imply $x_1 = x_2$

Also, range of f(x) $\neq$ co-domain of f(x)

Because 1 $\in R$ should also be image of some x $\in R$ then

$$\frac{x}{1+x^2}=1 \Rightarrow x^2-x+1=0$$; which has no real roots.

6. Summary

- Let A and B be two non-empty sets. A relation R from A to B is said to be a function if for each a ∈ A, there exists a unique y ∈ B such that y = f(x).
- Total number of functions from A to B = m^n , where n(A) = m and n(B) = n
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- A function $f : A \to B$ is called a real valued function if B is a subset of R, set of real numbers, and if both A and B are subsets of R then f is a real function.
- A function $f : A \to B$ is said to be injective or one-one function if each element of A has unique image in B, i.e, no two ‘a’ have same image.
- $f : A \to B$ is one-one ⇔ f(a) = f(b) ⇒ a = b for all a, b ∈ A
- If f is not one-one function then it is called many-one function.
- A function $f : A \to B$ is said to be onto or surjective if each element y ∈ B is image of some element of A under f.
- $f : A \to B$ is said to be onto or surjective ⇔ for each y ∈ B, there exists x ∈ A such that y = f(x)
- A function f(x) is onto if range = co-domain
- If a function is not onto, it is called into function.
- A function $f : A \to B$ is said to be bijective if f is one-one and onto function.