1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 03 (Class XII, Semester - 1)
Module Name/Title	Equivalence Relation - Part 1
Module Id	lemh_10101
Pre-requisites	Cartesian product of sets, Relations, Domain, Co-domain, Range
Objectives	After going through this lesson, the learners will be able to understand the following: • Understand reflexive relations. • Understand symmetric relations. • Understand transitive relations. • Understand equivalence relations. • Apply equivalence relations to solve the problems.
Keywords	Relations, Domain, Range

2. Development Team

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Table of Contents :

1. Introduction :

In class XI, you have already studied basics of relations and functions. In this chapter, you will study about various types of relations and functions and their properties. Let's revise previous concepts before starting new concepts.

2. Relations :

A **relation** R from a non-empty set A to another non-empty set B is an improper subset of A x B.

Here, we can recall that **A x B is Cartesian product of sets** A and B and defined as

$$A \times B = \{(a,b) : a \in A, b \in B\}$$

i.e., R is a relation from A to B $\Leftrightarrow R : A \to B \wedge R \subseteq A \times B$

$(a,b) \in R \Leftrightarrow a\,R\,b \wedge (a,b) \notin R \Leftrightarrow a \, is \, not \, related \, b$

If n(A) = m and n(B) = n then n(A x B) = m x n

Number of relations possible from A to B = Number of subsets of A x B = 2^{mn}

Since, R = {(a, b): a R b}

So, set of all values of 'a' is called **domain** and set of all values of 'b' is called **range** of R.

i.e., Domain of R = {a} $\subseteq$ A

Range of R = {b} $\subseteq$ B

Co-domain of R = B

Clearly, Range $\subseteq$ Co-Domain

For example, let R be a relation from A to B i.e., $R : A \to B$ where A = {1, 2, 3} and B = {2, 3, 5} and defined as R = $\{(a,b) : b = a+1, a \in A, b \in B\}$ then

R = {(1, 2), (2, 3)} $\subseteq A \times B$

Domain of R = {1, 2} $\subseteq A$

Range of R = {2, 3} $\subseteq B$

3. Trivial Relations :

Let A be a non-empty set and R is a relation defined on A, i.e., $R \subseteq A \times A$

(i) **Empty / Void / Null Relation –** Since $\varphi \subset A \times A$, therefore, φ is a null relation on A.

e.g. Let A = {1, 2, 3} and R = {(a, b): a + b = 7; a, b $\in$ A}

then R = { } as a + b $\neq$ 7 for any element of A.

(ii) **Universal Relation –** Since, $A \times A \subset A \times A$, we can say this is universal relation on A.

e.g. Let R be a relation defined on set of integers such that $R : Z \rightarrow Z$ and

R = $\{(a, b) : |a - b| \in Z, \forall a, b \in Z\}$ then R = Z and hence universal relation.

Both Empty relation and Universal relation are called **Trivial Relations**.

4. Reflexive Relation :

A relation R on set A is **reflexive** if every element of A is related to itself.

A relation $R : A \rightarrow A$ is reflexive $\Leftrightarrow$ (a, a) $\in$ A for all a $\in$ A

Example 1: Check if the relations R = {(1, 1), (1, 2), (2, 2), (3, 3)} and R_1 = {(1, 2), (2, 2), (3, 2)} defined on set A = {1, 2, 3} are reflexive.

Solution: Since (a, a) $\in$ R for all a $\in$ A for relation R, therefore, R is reflexive.

Since (1, 1) $\notin$ R_1, therefore, R_1 is not reflexive.

Example 2: Check if the relation R defined on set A of all lines in a plane as R = {(L_1, L_2): L_1 is parallel to L_2} is reflexive.

Solution: Since every line L_1 is considered to be parallel to it, therefore, R is reflexive.

Example 3: Check if the relation R defined on set A of all lines in a plane as R = {(L_1, L_2): L_1 is perpendicular to L_2} is reflexive.

Solution: Since no line L_1 can be perpendicular to it, therefore, R is not reflexive.

5. Symmetric Relation :

A relation R on set A is **symmetric** if $(a, b) \in A \Rightarrow (b, a) \in R$ for all $a, b \in A$

Example 4: Check if the relations R = {(1, 1), (1, 2), (2, 2), (3, 3), (2, 1)} and R_1 = {(1, 2), (2, 2), (3, 2), (2, 3)} defined on set A = {1, 2, 3} are symmetric.

Solution: Since $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$ for relation R, therefore, R is reflexive.

Since $(1, 2) \in R$ but $(2, 1) \notin R_1$, therefore, R_1 is not symmetric.

Example 5: Check if the relation R defined on set A of all lines in a plane as R = {(L_1, L_2): L_1 is parallel to L_2} is symmetric.

Solution: Let $(L_1, L_2) \in R \Rightarrow L_1 \parallel L_2$

$\Rightarrow L_2 \parallel L_1$

$\Rightarrow (L_2, L_1) \in R$

Thus, R is symmetric relation.

Example 6: Check if the relation R defined on set A of all lines in a plane as R = {(L_1, L_2): L_1 is perpendicular to L_2} is symmetric.

Solution: Let $(L_1, L_2) \in R \Rightarrow L_1 \perp L_2$

$\Rightarrow L_2 \perp L_1$

$\Rightarrow (L_2, L_1) \in R$

Thus, R is symmetric relation.

6. Transitive Relation :

A relation R on set A is **transitive** if $(a, b), (b, c) \in A \Rightarrow (a, c) \in R$ for all $a, b, c \in A$

Example 7: Check if the relations R = {(1, 1), (1, 2), (2, 3), (3, 3)} and R_1 = {(2, 1), (2, 3), (3, 1)} defined on set A = {1, 2, 3} are reflexive.

Solution: Since $(1, 2), (2, 3) \in R$ but $(1, 3) \notin R$, therefore, R is not transitive.

Since (2, 3), (3, 1) $\in$ R and (2, 1) $\in$ R, therefore, R is transitive.

Example 8: Check if the relation R defined on set A of all lines in a plane as R = {(L_1, L_2): L_1 is parallel to L_2} is transitive.

Solution: Let (L_1, L_2), (L_2, L_3) $\in$ R $\Rightarrow L_1 \parallel L_2$ and $L_2 \parallel L_3$

$\Rightarrow L_1 \parallel L_3$

$\Rightarrow$ (L_1, L_3) $\in$ R

Thus, R is transitive relation.

Example 9: Check if the relation R defined on set A of all lines in a plane as R = {(L_1, L_2): L_1 is perpendicular to L_2} is transitive.

Solution: Let (L_1, L_2), (L_2, L_3) $\in$ R $\Rightarrow L_1 \perp L_2$ and $L_2 \perp L_3$

$\Rightarrow$ L_1 is not perpendicular to L_3

$\Rightarrow$ (L_1, L_3) $\notin$ R

Thus, R is not transitive relation.

7. Equivalence Relation :

A relation R on set A is **equivalence relation** if R is reflexive, symmetric and transitive.

An equivalence relation R defined on a non-empty set A divides the set A into pair-wise disjoint subsets, which are called equivalence classes.

[a] denotes the equivalence class of a $\in$ A under the relation R.

We can say that [a] = { x: x $\in$ A and x R a}

We will learn these concepts with the help of following examples.

Example 10: Check is the relation R defined on set of integers Z as R = {(x, y): x – y is an integer} an equivalence relation.

[NCERT Exemplar]

Solution: Given relation R: Z $\rightarrow$ Z defined as R = {(x, y): x – y is an integer}

Since, x – x = 0 $\in$ Z $\Rightarrow$ (x, x) $\in$ R $\Rightarrow$ R is reflexive.

Let (x, y) $\in$ R $\Rightarrow$ x – y $\in$ Z

$\Rightarrow \quad -(y-x) \in Z$

$\Rightarrow \quad (y-x) \in Z$

$\Rightarrow \quad (y, x) \in R$

$\Rightarrow$ R is symmetric.

Let $(x, y), (y, z) \in R \Rightarrow x-y \in Z$ and $y-z \in Z$

$\Rightarrow \quad (x-y)+(y-z) \in Z$

$\Rightarrow \quad (x-z) \in Z$

$\Rightarrow \quad (x, z) \in R$

$\Rightarrow$ R is transitive.

Thus R is an equivalence relation.

Example 11: Let R be a relation defined on set of integers Z as R = {(a, b): 2 divides a – b}. Show that R is an equivalence relation. Also write equivalence class of 0.

[NCERT Exemplar]

Solution: Given relation R: Z $\rightarrow$ Z defined as R = {(x, y): 2 divides x – y}

Since, $x - x = 0$, which is divisible by 2 $\Rightarrow (x, x) \in R \Rightarrow$ R is reflexive.

Let $(x, y) \in R \Rightarrow x-y$ is divisible by 2

$\Rightarrow \quad -(y-x)$ is also divisible by 2

$\Rightarrow \quad (y-x)$ is divisible by 2

$\Rightarrow \quad (y, x) \in R$

$\Rightarrow$ R is symmetric.

Let $(x, y), (y, z) \in R \Rightarrow x-y$ is divisible by 2 and $y-z$ is divisible by 2

$\Rightarrow \quad (x-y)+(y-z)$ is divisible by 2

$\Rightarrow \quad (x-z)$ is divisible by 2

$\Rightarrow \quad (x, z) \in R$

$\Rightarrow$ R is transitive.

Thus R is an equivalence relation.

Equivalence class of 0 = [0] = set of elements related to 0

$[0]=\{0, \pm 2, \pm 4, \pm 6, \ldots\ldots.\}$

Example 12: Show that the relation R defined on set of real numbers R as R = {(a, b): $a \le b^2$ } is neither reflexive nor symmetric nor transitive.

[NCERT Exemplar]

Solution: Since $\frac{1}{2} \not\le \left(\frac{1}{2}\right)^2 \Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right) \notin R$ $\Rightarrow$ R is not reflexive.

(1, 2) $\in$ R as $1 < 2^2$ but $2 \not\le 1^2$ $\Rightarrow (2, 1) \notin R$ $\Rightarrow$ R is not symmetric

Since (5, 3), (3, 2) $\in$ R as $5 < 3^2$ and $3 < 2^2$

But $5 \not\le 2^2$ $\Rightarrow (5, 2) \notin R$ $\Rightarrow$ R is not transitive

Example 13: Show that the relation R defined on the set A of all the triangles as $R = \{(T_1, T_2) : T_1 \sim T_2 ; T_1, T_2 \in A\}$ is an equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5; T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles out of these three are related?

[NCERT Exemplar]

Solution: Since $T_1 \sim T_1 \Rightarrow$ $(T_1, T_1) \in R$ $\Rightarrow$ R is reflexive

Let $(T_1, T_2) \in R$ $\Rightarrow T_1 \sim T_2$

$\Rightarrow T_2 \sim T_1$

$\Rightarrow (T_2, T_1) \in R$

Thus R is symmetric.

Let $(T_1, T_2), (T_2, T_3) \in R$ $\Rightarrow T_1 \sim T_2$ and $T_2 \sim T_3$

$\Rightarrow T_1 \sim T_3$

$\Rightarrow (T_1, T_3) \in R$

Thus R is transitive.

Hence R is an equivalence relation.

Since sides of triangles T_1 and T_3 are proportional, therefore, T_1 and T_3 are related.

Example 14: Let A = {1, 2, 3,…, 9} and R be the relation defined on A x A as $R = \{(a, b) R (c, d) \Leftrightarrow a + d = b + c ; \forall (a, b), (c, d) \in A \times A\}$. Show that R is an equivalence relation. Also obtain equivalence class [(2, 5)].

[NCERT Exemplar]

Solution: Since a + b = b + a $\Longrightarrow$ (a, b) R (a, b) $\Longrightarrow$ R is reflexive

Let $(a,b)\,R\,(c,d) \Longrightarrow a+d=b+c$

$\Longrightarrow$ c + b = a + d

$\Longrightarrow$ c + b = d + a

$\Longrightarrow$ (c,d) R (a, b)

Thus R is symmetric.

Let (a, b) R (c, d) and (c, d) R (e, f) $\Longrightarrow$ a + d = b + c and c + f = d + e

$\Longrightarrow$ a + d + c + f = b + c + d + e (adding both the equations)

$\Longrightarrow$ a + f = b + e

$\Longrightarrow$ (a, b) R (e, f)

Thus R is transitive.

Hence R is an equivalence relation.

[(2, 5)] = {(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)}

8. Summary :

- A relation R from a non-empty set A to another non-empty set B is an improper subset of A x B.
- R is a relation from A to B $\Leftrightarrow R: A \rightarrow B \wedge R \subseteq A \times B$
- Number of relations possible from A to B = Number of subsets of A x B = 2^{mn}, where n(A) = m and n(B) = n
- Since, $\varphi \subset A \times A$, therefore, φ is a null relation on A.
- Since, $A \times A \subset A \times A$, we can say this is universal relation on A.
- Both Empty relation and Universal relation are called Trivial Relations.
- A relation R on set A is reflexive if every element of A is related to itself. A relation $R: A \rightarrow A$ is reflexive $\Leftrightarrow$ (a, a) $\in$ A for all a $\in$ A
- A relation R on set A is symmetric if (a, b) $\in$ A $\Rightarrow$ (b, a) $\in$ R for all a, b $\in$ A
- A relation R on set A is transitive if (a, b), (b, c) $\in$ A $\Rightarrow$ (a, c) $\in$ R for all a, b, c $\in$ A
- A relation R on set A is equivalence relation if R is reflexive, symmetric and transitive.

- An equivalence relation R defined on a non-empty set A divides the set A into pair-wise disjoint subsets, which are called equivalence classes.
- [a] denotes the equivalence class of a $\in$ A under the relation R.
- $[a] = \{ x : x \in A \text{ and } x \, R \, a \}$