1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 02 (Class XI, Semester - 2)
Module Name/Title	Mathematical Reasoning- Statements and its types, Words and Phrases- Part 2
Module Id	kemh_21402
Pre-requisites	1. Some basic ideas of Mathematical Reasoning 2. Some special words and phrases (Connectives and Quantifiers) 3. Inclusive 'or', exclusive 'or'
Objectives	After going through this module, the learners will be able to understand the followings: 1. Conditional statements 2. Contrapositive and converse 3. Validating Statements
Keywords	Implications, Contrapositive, Converse, Validating Statement

2. Development Team

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Introduction:

In the previous module, we have learnt some basic ideas of Mathematical Reasoning. The basic unit involved in mathematical reasoning is a mathematical statement. The collection of words is known as sentence and a sentence is called a mathematically acceptable statement if it is either true or false but not both simultaneously. We have also learnt some special words and phrases, which are respectively known as connectives and quantifiers.

In this module, we will learn when a statement is said to be true or false. We will learn some techniques to find when a statement is valid and learn some general rules for checking whether a statement is true or not.

Implications:

The conditional statement:

If p and q are any two statements, then the compound statement "if p then q" formed by joining p and q by a connective 'if then' is called a conditional statement or an implication and is written in symbolic form as $p \rightarrow q$ or $p \Rightarrow q$. Here, p is called hypothesis (or antecedent) and q is called conclusion (or consequent) of the conditional statement (p ⇒ q).

Observe that the conditional statement p → q reflects the idea that whenever it is known that p is true, it will have to follow that q is also true.

Example: 1

Write the conditional statement, $p \rightarrow q$, where

p : It is raining today.

q : $2 + 3 > 4$

Solution:

The required conditional statement is,

"If it is raining today, then 2 + 3 > 4"

Let us first discuss the implications;

"if-then", "only if" and "if and only if"

The statements with "if-then" are very common in mathematics.

For example, the statement,

r: If you are born in some country, then you are a citizen of that country.

In this statement, we observe that it has two simple statements *p* and *q* given by;

p: you are born in some country.
q: you are citizen of that country.

Then the sentence "if *p* then *q*" says that in the event if *p* is true, then *q* must be true.
The most important facts about the sentence "if *p* then *q*" is that it does not say anything on *q* when *p* is false.

For example, if you are not born in the country, then you cannot say anything about *q*. In other words" not happening of *p* has no effect on happening of *q*.

Another point to be noted for the statement "if *p* then *q*" is that the statement does not imply that *p* happens. There are several ways of understanding "if *p* then *q*" statements. Let us understand it with the following statement;

r: If a number is a multiple of 9, then it is a multiple of 3.

Let *p* and *q* be the component statements,

p : a number is a multiple of 9.
q: a number is a multiple of 3.

Then, if *p* then *q* is the same as the followings:

1. *p* implies *q* is denoted by $p \Rightarrow q$.
The symbol $\Rightarrow$ stands for implies.
This says that a number is a multiple of 9 implies that it is a multiple of 3.

2. *p* is a sufficient condition for *q*.
This says that knowing that a number as a multiple of 9 is sufficient to conclude that it is a multiple of 3.

3. *p* only if *q*.
This says that a number is a multiple of 9 only if it is a multiple of 3.

4. *q* is a necessary condition for *p*.
This says that when a number is a multiple of 9, it is necessarily a multiple of 3.

5. **$\sim q$ implies $\sim p$.**

This says that if a number is not a multiple of 3, then it is not a multiple of 9.

Let us now discuss how statements can be formed from a given statement with "if-then".

Contrapositive and converse:

Contrapositive and converse are certain other statements which can be formed from a given statement with "if-then". For example, let us consider the following "if-then" statement.

p: If the physical environment changes, then the biological environment changes.
Then the contrapositive of this statement is;

q: If the biological environment does not change, then the physical environment does not change.

Note that both these statements convey the same meaning.

Let us understand it with the help of some more examples.

Example: 2

Write the contrapositive of the following statements:

(i) If you are born in India, then you are a citizen of India.

(ii) If a triangle is equilateral, it is isosceles.

(iii) If my car is in the repair shop, then I cannot go to the market.

Solution:

The contrapositive of the given statements are;

(i) If you are not a citizen of India, then you were not born in India.

(ii) If a triangle is not isosceles, then it is not equilateral.

(iii) "If I can go to the market, then my car is not in the repair shop".

What do you observe in the above examples?

These examples show that the contrapositive of the statement "if p, then q" is "if $\sim q$, then $\sim p$".

The given statement in symbolic form is, "$p \rightarrow q$". Therefore, its contrapositive is given by, "$\sim q \rightarrow \sim p$".

Let us now discuss another term called "**Converse**".

The converse of a given statement "if p, then q" is " if q, then p ".

The conditional statement "q → p" is called the converse of the conditional statement " p → q "

Let us take a statement to understand it,

p: If a number is divisible by 10, it is divisible by 5.

Then its converse is,

q: If a number is divisible by 5, then it is divisible by 10.

Let us take an example and learn to write converse of a given statement.

Example: 3

Write the converse of the following statements;

(i) If ABC is an equilateral triangle, then ABC is an isosceles triangle (ii) If a number n is even, then n^2 is even.
(iii) If two integers a and b are such that a > b, then a – b is always a positive integer.

Solution:

The converse of the given statements are;

(i)"If ABC is an isosceles triangle, then ABC is an equilateral triangle."

(ii) If a number n^2 is even, then n is even.

(iii) If two integers a and b are such that a – b is always a positive integer, then a > b.

Example: 4

For each of the following "if-then" statements, identify the corresponding component statements and then check whether the statements are true or not.

(i) If a triangle ABC is equilateral, then it is isosceles.

(ii) If a and b are integers, then ab is a rational number.

Solution:

(i)The component statements are;

p : Triangle ABC is equilateral.

q : Triangle ABC is isosceles.

Since an equilateral triangle is isosceles, we infer that the given compound statement is true.

(ii) The component statements are;

p : a and b are integers.

q : ab is a rational number.

Since the product of two integers is an integer and therefore a rational number, the compound statement is true.

The biconditional statement:

If two statements p and q are connected by the connective, "if and only if" then the resulting compound statement, "p if and only if q" is called a biconditional of p and q and is written in symbolic form as $p \leftrightarrow q$ or $p \Leftrightarrow q$.

'If and only if', represented by the symbol " $\leftrightarrow$" or "$\Leftrightarrow$" means the following equivalent forms for the given statements p and q.

(i) p if and only if q
(ii) q if and only if p
(iii) p is necessary and sufficient condition for q and vice-versa

(iv) $p \Leftrightarrow q$

Example: 5

Given below are two pairs of statements. Combine these two statements to form the biconditional statement using "if and only if ".

(i) p : One is less than seven

q : Two is less than eight

(ii) p: If the sum of digits of a number is divisible by 3, then the number is divisible by 3.

q: If a number is divisible by 3, then the sum of its digits is divisible by 3.

(iii) p: If a rectangle is a square, then all its four sides are equal.

q: If all the four sides of a rectangle are equal, then the rectangle is a square.

(iv) p: Today is 14th of August.

q : Tomorrow is Independence day.

Solution:

The biconditional statements ($p \leftrightarrow q$) from the given simple statements p and q are;

(i) "One is less than seven, if and only if two is less than eight".
(ii) A number is divisible by 3 if and only if the sum of its digits is divisible by 3.
(iii) A rectangle is a square if and only if all its four sides are equal.
(iv) "Today is 14th of August if and only if tomorrow is Independence Day".

Example: 6

Translate the following biconditional statement into symbolic form:

"ABC is an equilateral triangle if and only if it is equiangular".

Solution:

The simple statements of the given biconditional statement are;

p: ABC is an equilateral triangle and

q: ABC is an equiangular triangle.

Then, the given statement in symbolic form is given by,

$$\mathrm{p} \leftrightarrow \mathrm{q} \text{ or } p \Leftrightarrow q.$$

Validating Statements:

Validity of a statement means checking when a given statement is true and when it is not true. This depends upon which of the connectives, quantifiers and implication is being used in the statement.

Let us think on some questions:
What does the statement mean? What would it mean to say that this statement is true and when this statement is not true?

Obviously the answer to these questions depend upon which of the special words and phrases "and", "or", which of the implications "if-then", "if and only", and which of the quantifiers "for every", "there exists", appear in the given statement.

Let us discuss some techniques to find when a statement is valid and list some general rules for checking whether a statement is true or not.

Rule: 1

(Validity of statement with 'And')

If p and q are mathematical statements, then in order to show that the statement "p and q" is true, the following steps are followed;

Step-1: Show that the statement p is true.

Step-2: Show that the statement q is true.

Example: 7

Check the validity of the statement;

(i) "100 is a multiple of 4 and 5".
(ii) Every rectangle is a square and every square is a rectangle.
(iii) Delhi is the capital of India and Islamabad is the capital of Pakistan.

Solution:

(i) The component statements of the given statement are;

p: "100 is a multiple of 4".

q: "100 is a multiple of 5".

Connected by the connector "And". Both the component statements are true. Hence by the above Rule-1, the given statement is true.

(ii) The component statements of the given statement are;

p: "Every rectangle is a square".

q: "Every square is a rectangle".

Connected by the connector "And". According to Rule-1, both the component statements should be true for the given statement to be true. Here the first component statement p is false and the second component statement q is true, thus by Rule-2, the given statement is false.

(iii)The component statements of the given statement are;

p: "Delhi is the capital of India".

q: "Islamabad is the capital of Pakistan".

Connected by the connector "And". In this case also both the component statements are true. Hence the given statement is true.

Rule: 2

(Validity of Statements with "Or")

If *p* and *q* are mathematical statements, then in order to show that the statement "*p* or *q*" is true, one must do the following;

Show either statement 'p' is true or statement 'q' is true.

For this following steps can be followed,

Step-1: By assuming that *p* is false, show *q* must be true.

Step-2: By assuming that *q* is false, show *p* must be true.

Example: 8

Check the validity of the statement;

(i) "60 is a multiple of 3 or 5".
(ii) 2 + 4 = 6 or 2 + 4 = 7
(iii) 9 is a prime number or 9 is an even number.

Solution:

(i) The component statements of the given statement are;

p: "60 is a multiple of 3".

q: "60 is a multiple of 5".

These component statements are connected by the connector "Or". Both the component statements are true. Hence by Rule-2, the given statement is true.

(ii) The component statements of the statement are;

p: 2 + 4 = 6.

q: 2 + 4 = 7.

The component statements are connected by the connector "Or".
In this case the first component statement *p* is true and the second component statement *q* is false. By Rule-2, the given statement is true.

(iii) The component statements of the statement are;

p: 9 is a prime number.

q: 9 is an even number.

The component statements are connected by the connector "Or".
In this case the both the component statements p and q are false. Hence the given statement is false.

Rule: 3

(Validity of Statements with "If-then")

In order to prove the statement "if p then q" we need to show that any one of the following case is true.

Case-1

By assuming that p is true, prove that q must be true. (Direct method)

i.e., $p \Rightarrow q$

Case-2

By assuming that q is false, prove that p must be false. (Contrapositive method)

i.e., $\sim q \Rightarrow \sim p$.

Example: 9

Check whether the following statement is true or not.

If $x, y \in Z$ are such that x and y are odd, then xy is odd.

Solution:

Let the given statement in symbolic form be $p \Rightarrow q$, then

p: $x, y \in Z$ such that x and y are odd.

and q: xy is odd.

(i)To check the validity of the given statement, by Case 1 of Rule 3,
we assume that p is true and then we have to prove that q is true.
Statement p is true means x and y are odd integers.
So let,

$x = 2m + 1$, for some integer m.

and $y = 2n + 1$, for some integer n.

Then, $xy = (2m + 1)(2n + 1)$

$= 2(2mn + m + n) + 1$, which is odd.

This shows that statement "q: xy is odd", is true.

Hence, the given statement is true.

(ii)Suppose we want to check this by using Case 2 of Rule 3, then by assuming that q is false, we have to prove that p must be false.

i.e., $\sim q \Rightarrow \sim p$.

We proceed as follows;

Let us assume that q is not true. This implies that we consider the negation of the statement q to be true.

The negation of the statement q is;

$\sim q$: Product xy is even.

This is possible only if either x or y is even.

But according to statement p, both $x, y \in \mathbb{Z}$ are odd.

This shows that statement p is not true.

Hence, we get, $\sim q \Rightarrow \sim p$

Which shows that the given statement is true.

Note:

1) The above example illustrates that to prove $p \Rightarrow q$, it is enough to show $\sim q \Rightarrow \sim p$.

2) **$\sim q \Rightarrow \sim p$ is the contrapositive of the statement $p \Rightarrow q$**

Rule: 4

(Validity of Statements with "if and only if")

In order to prove the statement "p if and only if q", we need to show;

Step-1: If p is true, then q is true and

Step-2: If q is true, then p is true.

Example: 10

Check whether the following statement is true or not.
"The integer n is odd if and only if n^2 is odd".

Solution:

Let p and q denote the statements,

p: The integer n is odd.

q: n^2 is odd.

The given compound statement "p if and only if q" is biconditional of p and q ($p \leftrightarrow q$).

To check the validity of "*p* if and only if *q*", we need to follow two steps, (Rule-4**).**

Step-1: If p is true, then q is true and

Step-2: If q is true, then p is true.

Step-1: **(If *p*, then *q*)**

According to the given statement, "If p, then q" is;
"If the integer n is odd, then n^2 is odd".
We have to check whether this statement is true or not.
Let us assume that n is odd.
Then $n = 2k + 1$ when k is an integer.
Thus, $n^2 = (2k + 1)^2$
$= 4k^2 + 4k + 1$ (which is odd)
Since, n^2 is one more than an even number, hence it is odd.
$\therefore$ n is odd integer $\Rightarrow$ n^2 is odd integer
$\therefore$ The statement, "If p, then q" is true.

Step-2: **(If *q*, then *p*)**

The statement, "If q, then p" is;
"If n is an integer and n^2 is odd, then n is odd".
We have to check whether this statement is true or not.
We will check this by contrapositive method.
The contrapositive of the statement; "If n is an integer and n^2 is odd, then n is odd", is;
"If n is not an odd integer, then n^2 is not an odd integer".
The above statement is equivalent to the statement;
"If n is an even integer, then n^2 is an even integer".
So, let n = 2k for some k, where k is an integer (n is not odd).
Then, $n^2 = 4k^2$, which is even.
Therefore, n^2 is an even integer.
$\Rightarrow$ n^2 is not an odd integer.
Hence we got,
n is not odd integer $\Rightarrow$ n^2 is not odd integer
$\therefore$ The statement, "If q, then p" is true.
Combining the two (Rule-4**)**,
"*p* if and only if *q*" is true.
Hence, the statement,
"The integer n is odd if and only if n^2 is odd", is true.

Example: 11

By giving a counter example, show that the following statement is false.

"If n is an odd integer, then n is prime".

Solution:

The given statement is in the form, "if p then q", Where,

p: n is an odd integer.

q: n is prime.

We have to find a counter example to show that the given statement is false. For this purpose we have to find a counter example to show,

"if p then $\sim q$"

Hence, we have to look for an odd integer n which is not a prime number.

Consider the number "9", it is an odd integer but not a prime number. So, "$n = 9$" is a counter example.

Hence, we conclude that the given statement is false.

Note:

In mathematics, counter examples are used to disprove a statement. However, generating many examples in favour of a statement will never provide validity of the statement.

Summary:

In this module we have learnt,

1) The meaning of implications "If", "only if", "if and only if".

2) A sentence with "if p, then q" can be written in the following ways;
   (i) p implies q (denoted by $p \Rightarrow q$)
   (ii) p is a sufficient condition for q
   (iii) q is a necessary condition for p
   (iv) p only if q
   (v) $\sim q$ implies $\sim p$

3) The contrapositive of a statement $p \Rightarrow q$ is the statement $\sim q \Rightarrow \sim p$

4) The converse of a statement $p \Rightarrow q$ is the statement $q \Rightarrow p$.

5) $p \Rightarrow q$ together with its converse $q \Rightarrow p$, gives "p if and only if q".

6) The following methods are used to check the validity of statements
   (i) Direct method
   (ii) Contrapositive method
   (iii) Method of contradiction
   (iv) Using a counter example.