1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 02 (Class XI, Semester - 2)
Module Name/Title	Limits – Part 2
Module Id	Kemh_21302
Pre-requisites	Knowledge about Limits, LHL and RHL
Objectives	After going through this lesson, the learners will be able to understand the following: 1. Limits of Trigonometric functions 2. Limits of exponential and logarithmic function
Keywords	Trigonometric limits, exponential and logarithmic limits

2. Development Team

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Introduction

We know the concept of limits of a function is crucial in understanding the concept of derivative. In the previous module on limits, we have already studied about concept of limits (left hand limit and right hand limit), algebra of limits, limits of polynomial and rational functions. In this module, we will be studying about limits of trigonometric functions, exponential functions and logarithmic functions.

Before heading towards these limits, let us first recapitulate in brief what we have studied in previous module.

> Facts from history
>
> - Before 1900, it was thought that calculus is quite difficult to teach. So, calculus become beyond the reach of youngsters. But just in 1900, John Perry and others in England started propagating the view calculus could be taught in schools. F.L. Griffin, pioneered the teaching of calculus to first year students. This was considered to be the most daring act in those days.
> - Today not only the mathematics but many other subjects such as Physics, Chemistry, Economics and Biological

Recapitulation:

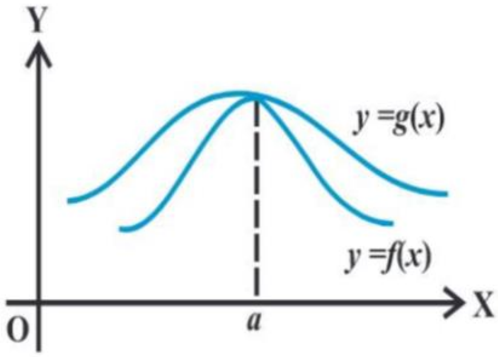
- **Left hand limit (LHL):** The expected value of the function as dictated by the points to the left of a point defines the left hand limit of the function at that point.
- **Right hand limit (RHL):** The expected value of the function as dictated by the points to the right of a point defines the right hand limit of the function at that point.
- **Limit:** Limit of a function at a point is the common value of the left and right hand limits, if they coincide.
- **Notation:**
  - Limit: $\lim_{x \to a} f(x)$
  - LHL: $\lim_{x \to a^-} f(x)$
  - RHL: $\lim_{x \to a^+} f(x)$
- **Algebra of Limits:** For functions f and g the following holds:
  - $\lim_{x \to a}(f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
  - $\lim_{x \to a}(f(x).\, g(x)) = \lim_{x \to a} f(x).\, \lim_{x \to a} g(x)$
  - $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$
- **Some standard limits**
  - $\lim_{x \to a} k = k$ $\qquad$ $where\ k\ is\ any\ real\ fixed\ number$
  - $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

Limits of Trigonometric functions

In order to evaluate trigonometric limits we will be using following:

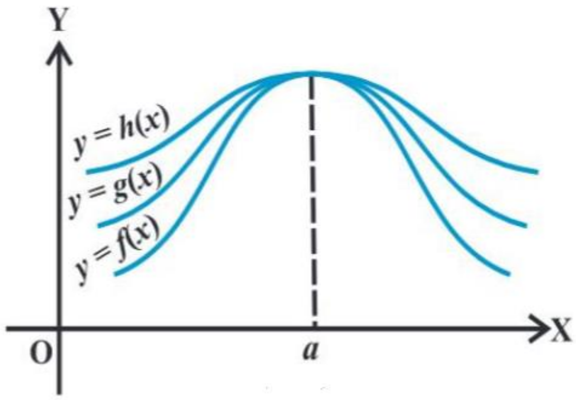
(I)

Theorem Let f and g be two real valued functions with the same domain such that $f(x) \le g(x)$ for all x in the domain of definition, For some a, if both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$. This is illustrated in **Figure below**



(II)

Theorem (Sandwich Theorem) Let f, g and h be real functions such that $f(x) \le g(x) \le h(x)$ for all x in the common domain of definition. For some real number a, if $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = l$. This is illustrated in **Figure below**



(III)

$$\cos x < \frac{\sin x}{x} < 1 \quad \text{for } 0 < |x| < \frac{\pi}{2}$$

Proof We know that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$. Hence, it is sufficient to prove the inequality for $0 < x < \frac{\pi}{2}$.

In the Fig 13.10, O is the centre of the unit circle such that the angle AOC is x radians and $0 < x < \frac{\pi}{2}$. Line segments B A and CD are perpendiculars to OA. Further, join AC. Then

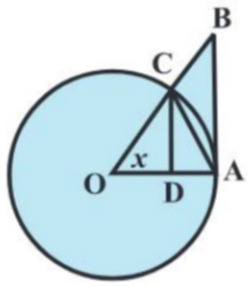


Fig 13.10

Area of ΔOAC < Area of sector OAC < Area of Δ OAB.

i.e., $\frac{1}{2}\text{OA.CD} < \frac{x}{2\pi}.\pi.(\text{OA})^2 < \frac{1}{2}\text{OA.AB}$.

i.e., CD < x . OA < AB.

From Δ OCD,

$\sin x = \frac{\text{CD}}{\text{OA}}$ (since OC = OA) and hence CD = OA $\sin x$. Also $\tan x = \frac{\text{AB}}{\text{OA}}$ and

hence AB = OA. $\tan x$. Thus

OA $\sin x$ < OA. x < OA. $\tan x$.

Since length OA is positive, we have

$\sin x < x < \tan x$.

Since $0 < x < \frac{\pi}{2}$, $\sin x$ is positive and thus by dividing throughout by $\sin x$, we have

$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$. Taking reciprocals throughout, we have

$\cos x < \frac{\sin x}{x} < 1$

which complete the proof.

With the help of above facts and theorem let us prove some standard and important limits given below:

- $\lim_{x\to 0}\frac{\sin x}{x} = 1$
- $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$
- $\lim_{x\to 0}\frac{\tan x}{x} = 1$

Solution: from (III) above we know that the function $\frac{\sin x}{x}$ is sandwiched between the functions $\cos x$ and constant function 1

That is $\cos x < \frac{\sin x}{x} < 1$

Therefore from (II) we get,

$$\lim_{x\to 0}\cos x < \lim_{x\to 0}\frac{\sin x}{x} < \lim_{x\to 0} 1$$

$\Longrightarrow \lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad as\ \left(\lim_{x\to 0}\cos x = 1\ and\ \lim_{x\to 0} 1 = 1\right)$

- Now, we will show $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{2\sin^2(x/2)}{x} = \lim_{x\to 0}\frac{2\sin x/2}{(x/2)} \times \sin x/2$$

$$= \lim_{x\to 0}\frac{2\sin x/2}{(x/2)} \times \lim_{x\to 0}\sin x/2 = 1$$

(Recall $1-\cos x = 2\sin^2(x/2)$)

Hence, $\boldsymbol{\lim_{x\to 0}\frac{1-\cos x}{x} = 0}$

Now, solve $\boldsymbol{\lim_{x\to 0}\frac{\tan x}{x} = 1}$. Isn't it easy? Direct application of $\lim_{x\to 0}\frac{\sin x}{x} = 1$. See below!

$$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sin x}{x\cos x} = \lim_{x\to 0}\frac{\sin x}{x}.\lim_{x\to 0}\frac{1}{\cos x} = 1$$

Similarly, you can evaluate more.

Let us take some examples.

Example: Evaluate $\boldsymbol{\lim_{x\to 0}\frac{\sin^2 19x}{x^2}}$

Solution: We have,

$$\lim_{x\to 0}\frac{\sin^2 x}{x^2} = \lim_{x\to 0}\frac{\sin 19x}{19x} \times \frac{\sin 19x}{19x} \times 19^2$$

$$\lim_{x\to 0}\frac{\sin^2 x}{x^2} = 19^2 \lim_{x\to 0}\frac{\sin 19x}{19x} \times \lim_{x\to 0}\frac{\sin 19x}{19x} \qquad (algebra\ of\ limits)$$

$$\lim_{x\to 0}\frac{\sin^2 x}{x^2} = 19^2 = 361$$

Example: Evaluate $\boldsymbol{\lim_{x\to 0}\frac{\sin ax}{\sin bx}}$

Solution: we have,

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx} = \lim_{x\to 0}\frac{\sin ax}{ax} \times \frac{bx}{\sin bx} \times \frac{ax}{bx}$$

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx} = \frac{a}{b}\lim_{x\to 0}\frac{\sin ax}{ax} \times \lim_{x\to 0}\frac{bx}{\sin bx} \qquad \left(as\ \lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx} = \frac{a}{b}$$

Example: Evaluate $\boldsymbol{\lim_{x\to 0}\frac{1-\cos 8x}{1-\cos 16x}}$

Solution: We have,

$$\lim_{x\to 0}\frac{1-\cos 8x}{1-\cos 16x} = \lim_{x\to 0}\frac{2\sin^2 4x}{2\sin^2 8x} \qquad (using\ \cos 2x = 1 - 2\sin^2 x)$$

$$\lim_{x\to 0}\frac{1-\cos 8x}{1-\cos 16x}=\lim_{x\to 0}\frac{\sin^2 4x}{\sin^2 8x}$$

$$\lim_{x\to 0}\frac{1-\cos 8x}{1-\cos 16x}=\lim_{x\to 0}(\frac{\sin 4x}{4x}\times\frac{\sin 4x}{4x}\times 16x^2)\ \lim_{x\to 0}(\frac{8x}{\sin 8x}\times\frac{8x}{\sin 8x}\times\frac{1}{64x^2})$$

$$\lim_{x\to 0}\frac{1-\cos 8x}{1-\cos 16x}=\frac{16x^2}{64x^2}=\frac{1}{4}$$

Example: $\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^2 x}$

Solution: we have,

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^2 x}=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{\sin^2 x}$$

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^2 x}=\lim_{x\to 0}\frac{\sin x-\sin x\cos x}{\sin^2 x\cos x}$$

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^2 x}=\lim_{x\to 0}\frac{\sin x\,(1-\cos x)}{\sin^2 x\cos x}$$

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^2 x}=\lim_{x\to 0}\frac{x}{\sin x}\times\frac{1-\cos x}{x}\times\frac{1}{\cos x}$$

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^2 x}=\lim_{x\to 0}\frac{x}{\sin x}\times\lim_{x\to 0}\frac{1-\cos x}{x}\times\lim_{x\to 0}\frac{1}{\cos x}$$

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^2 x}=0\qquad (as\ \lim_{x\to 0}\frac{x}{\sin x}=1,\lim_{x\to 0}\frac{1-\cos x}{x}=0,\lim_{x\to 0}\frac{1}{\cos x}=1\ )$$

Example: Evaluate $\lim_{y\to 0}\frac{(x+y)\sec(x+y)-x\sec x}{y}$

Solution: We have,

$$\lim_{y\to 0}\frac{(x+y)\sec(x+y)-x\sec x}{y}$$

$$=\lim_{y\to 0}\frac{x\sec(x+y)+y\sec(x+y)-x\sec x}{y}$$

$$=\lim_{y\to 0}\frac{x\,(\sec(x+y)-\sec x)+y\sec(x+y)}{y}$$

$$=\lim_{y\to 0}\frac{x\,(\sec(x+y)-\sec x)}{y}+\lim_{y\to 0}\ \sec(x+y)$$

$$=\lim_{y\to 0}\frac{x\,(\sec(x+y)-\sec x)}{y}+\lim_{y\to 0}\ \sec(x+y)$$

$$=\lim_{y\to 0}\frac{x(\cos x-\cos(x+y))}{y\cos x\cos(x+y)}+\lim_{y\to 0}\ \sec(x+y)$$

$$=\lim_{y\to 0}\frac{x}{\cos x\cos(x+y)}\times\lim_{y\to 0}\frac{2\sin(x+\frac{y}{2})\sin\frac{y}{2}}{2(\frac{y}{2})}+\lim_{y\to 0}\ \sec(x+y)$$

$$= \frac{x\sin x}{\cos x} \times \cos x + \sec x$$

$$= x\tan x\sec x + \sec x$$

Hence, $\lim_{y\to 0}\frac{(x+y)\sec(x+y) - x\sec x}{y} = x\tan x\sec x + \sec x$

It is interesting to note that in all above examples we have find the limits when variable approaches to zero (i.e., $x \to 0$). So, let us now see what happen when x tends to some other real number except zero.

Example: $\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$

Solution: We have,

$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$$

Let $(\pi - x) = y$

$y \to 0$ as $x \to \pi$

Implies

$$\lim_{y\to 0}\frac{\sin y}{\pi y} = \frac{1}{\pi}\lim_{y\to 0}\frac{\sin y}{y} = \frac{1}{\pi}$$

Hence,

$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)} = \frac{1}{\pi}$$

Example: Let $f(x) = \begin{cases} \frac{k(\sin x - \cos x)}{x-\pi/4} & x \neq \pi/4 \\ \sqrt{2} & x = \pi/4 \end{cases}$ and if $\lim_{x\to\pi/2} f(x) = f(\pi/4)$. Find the value of k

Solution: It is given that $\lim_{x\to\pi/2} f(x) = f(\frac{\pi}{4})$

Therefore, $\lim_{x\to\frac{\pi}{2}}\frac{k(\sin x - \cos x)}{x-\frac{\pi}{4}} = \sqrt{2}$

$$k\lim_{x\to\frac{\pi}{2}}\frac{(\sin x - \cos x)}{x-\frac{\pi}{4}} = \sqrt{2}$$

$$\Rightarrow k\lim_{x\to\frac{\pi}{2}}\frac{\sqrt{2}\left(\sin x\,(1/\sqrt{2}) - \cos x\,(1/\sqrt{2})\right)}{x-\frac{\pi}{4}} = \sqrt{2}$$

$$\Rightarrow k\lim_{x\to\frac{\pi}{2}}\frac{\sqrt{2}\left(\sin x\,(1/\sqrt{2}) - \cos x\,(1/\sqrt{2})\right)}{x-\frac{\pi}{4}} = \sqrt{2}$$

$$\Rightarrow k\lim_{x\to\frac{\pi}{2}}\frac{\sqrt{2}(\sin x\cos\pi/4 - \cos x\sin\pi/4)}{x-\frac{\pi}{4}} = \sqrt{2}$$

$$\Rightarrow k \lim_{x \to \frac{\pi}{2}} \frac{\sqrt{2} \sin\left(x - \frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \sqrt{2} \qquad as\ (\sin(x-y) = \sin x \cos y - \sin y \cos x)$$

$$\Rightarrow k \lim_{x \to \frac{\pi}{2}} \frac{\sin\left(x - \frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = 1 \qquad as\ (\lim_{x \to 0} \frac{\sin x}{x} = 1)$$

$\Rightarrow k = 1$

Hence, value of $k\ is\ equal\ to\ 1$

Example: $\lim_{x \to \pi/6} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x - 1}$

Solution: we have,

$$\lim_{x \to \pi/6} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x - 1} = \lim_{x \to \pi/6} \frac{(2\sin x - 1)(\sin x + 1)}{(2e \sin x - 1)(\sin x - 1)} = -3$$

Example: $\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos^2 x}}{\sin^2 x}$

Solution: we have,

When we put $x = 0$ the expression $\frac{\sqrt{2} - \sqrt{1 + \cos^2 x}}{\sin^2 x}$ takes form $\frac{0}{0}$

Therefore, on rationalizing the numerator we get

$$= \lim_{x \to 0} \frac{2 - (1 + \cos^2 x)}{\sin^2 x\, (\sqrt{2} - \sqrt{1 + \cos^2 x})}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{\sin^2 x\, (\sqrt{2} - \sqrt{1 + \cos^2 x})}$$

$$= \lim_{x \to 0} \frac{1}{(\sqrt{2} - \sqrt{1 + \cos^2 x})}$$

$$= \frac{1}{2\sqrt{2}}$$

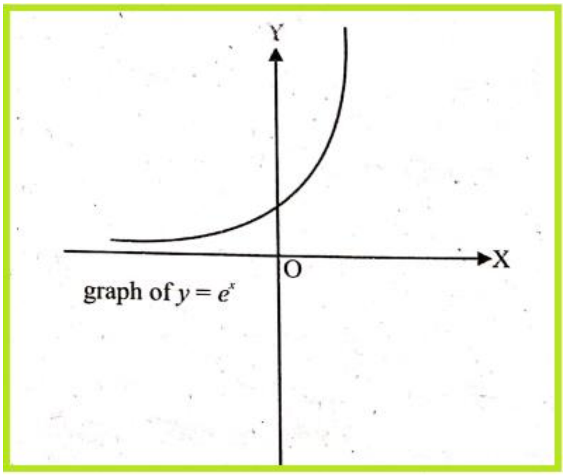
Hence, $\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos^2 x}}{\sin^2 x} = \frac{1}{2\sqrt{2}}$

In last example you must have noticed an exponential function but in a narrower view. Let us see these functions in a broader view.

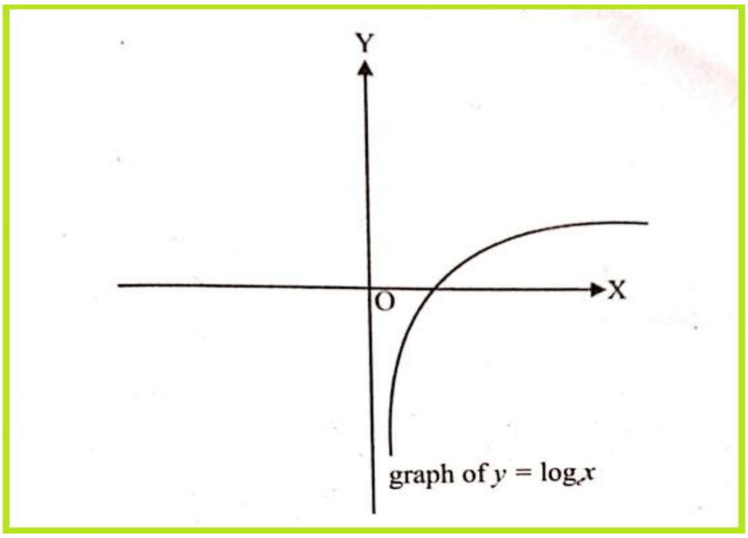
Limits of exponential and logarithmic functions

Before discussing evaluation of limits, let us introduce these two functions stating their domain, range and their graph.

Leonhard Euler (1707-1783), the great Swiss mathematician introduced the number e whose values lies between 2 and 3. This number is useful in defining exponential function and is defined as $f(x) = e^x, x \epsilon R$. Its domain is R, range is the set of positive real numbers. The graph of exponential function, i.e., $y = e^x$ is as given in figure below.



Similarly, the logarithmic function expressed as $\log_e R^+ \rightarrow R$ is given by$\log_e x = y$, if and only if $e^y = x$. Its domain is R^+ which is the set of all positive real numbers and range is R. The graph of logarithmic function $\mathrm{y} = \log_e x$ is shown in figure below:



Now, evaluate two of the important limits of these functions

Theorem: Prove that $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Proof: It will be interesting to note that

$$\frac{1}{1 + |x|} \leq \frac{e^x - 1}{x} \leq 1 + |x|(e - 2), x \in [-1,1] \sim \{0\}$$

Also, $\lim\limits_{x\to 0}\frac{1}{1+|x|}=\frac{1}{1+\lim\limits_{x\to 0}|x|}=1$

and $\lim\limits_{x\to 0}1+|x|(e-2)=1+(e-2)\lim\limits_{x\to 0}|x|=1+(e-2)0=1$

Therefore, by sandwich theorem, we get

$$\lim_{x\to 0}\frac{e^x-1}{x}=1$$

Also,

Theorem: Prove that $\lim\limits_{x\to 0}\frac{\log_e(1+x)}{x}=1$

Proof: let $\frac{\log_e(1+x)}{x}=\boldsymbol{y}$.

Then $\log_e(1+x)=xy$

$\Longrightarrow 1+x=e^{xy}$

$$\Longrightarrow \frac{e^{xy}-1}{x}=1$$

Or

$$\Longrightarrow \frac{e^{xy}-1}{xy}.y=1$$

$$\Longrightarrow \lim_{xy\to 0}\frac{e^{xy}-1}{xy}.\lim_{y\to 0}y=1 \qquad as\ x\longrightarrow 0\ \ xy\longrightarrow 0$$

$$\Longrightarrow \lim_{y\longrightarrow 0}y=1 \qquad as \quad \lim_{x\to 0}\frac{e^x-1}{x}=1$$

$$\Longrightarrow \lim_{x\longrightarrow 0}\frac{\log_e(1+x)}{x}=1$$

Example: Evaluate $\lim\limits_{x\to 0}\frac{e^x-1}{\sqrt{1-cos\,2x}}$

Solution: we have

$$\lim_{x\to 0}\frac{e^x-1}{\sqrt{1-cos\,2x}}$$

- (Recall $\lim\limits_{x\to 0}\frac{e^x-1}{x}=1$ and $1-cos\,2x=2\sin^2 x$)

Therefore,

$$\lim_{x\to 0}\frac{e^x-1}{\sqrt{1-cos\,2x}}=\lim_{x\to 0}\frac{e^x-1}{x}\times\lim_{x\to 0}\frac{x}{\sqrt{2\sin^2 x}} \qquad (multipying\ and\ dividing\ by\ x\ )$$

$$\lim_{x\to 0}\frac{e^x-1}{\sqrt{1-cos\,2x}}=\lim_{x\to 0}\frac{e^x-1}{x}\times\lim_{x\to 0}\frac{x}{\sqrt{2}\,\sin x}=\frac{1}{\sqrt{2}}$$

Example: Evaluate $\lim\limits_{x\longrightarrow 0}\frac{\log_e(1+x^3)}{sin^3x}=1$

Solution: we have,

$$\lim_{x\longrightarrow 0}\frac{\log_e(1+x^3)}{sin^3x}$$

- (Recall $\lim_{x\to 0}\frac{\log_e(1+x)}{x}=1$ *and* $\lim_{x\to 0}\frac{\sin x}{x}=1$)

Now,

$$\lim_{x\longrightarrow 0}\frac{\log_e(1+x^3)}{sin^3x}=\lim_{x\longrightarrow 0}\frac{\log_e(1+x^3)}{x^3}.\lim_{x\longrightarrow 0}\frac{x^3}{sin^3x}=1$$

While evaluating all the above limits. We can notice the following rule is kept in mind always.

A general rule that needs to be kept in mind while evaluating limits is the following.

Say, given that the limit $\lim_{x\to a}\frac{f(x)}{g(x)}$ exists and we want to evaluate this. First we check the value of $f(a)$ and $g(a)$. If both are 0, then we see if we can get the factor which is causing the terms to vanish, i.e., see if we can write $f(x)=f_1(x)\,f_2(x)$ so that $f_1(a)=0$ and $f_2(a)\neq 0$. Similarly, we write $g(x)=g_1(x)\,g_2(x)$, where $g_1(a)=0$ and $g_2(a)\neq 0$. Cancel out the common factors from $f(x)$ and $g(x)$ (if possible) and write

$$\frac{f(x)}{g(x)}=\frac{p(x)}{q(x)}, \text{ where } q(x)\neq 0.$$

Before ending this module let us revise some standard limits that we have learn in this module

Summary

- Let f and g be two real valued functions with the same domain such that $f(x)\le g(x)$ for all x in the domain of definition, for some a, if both $\lim_{x\to a}f(x)$ and $\lim_{x\to a}g(x)$ exist, then $\lim_{x\to a}f(x)\le\lim_{x\to a}g(x)$
- Let f , g and h be real valued functions such that $f(x)\le g(x)\le h(x)$ for all x in the common domain of definition, for some real number a, if $\lim_{x\to a}f(x)=l=\lim_{x\to a}h(x)$, then $\lim_{x\to a}g(x)=l$
- $\lim_{x\to 0}\frac{sin\ x}{x}=1$
- $\lim_{x\to 0}\frac{1-cos\ x}{x}=0$
- $\lim_{x\to 0}\frac{tan\ x}{x}=1$
- $\lim_{x\to 0}\frac{e^x-1}{x}=1$
- $\lim_{x\to 0}\frac{\log_e(1+x)}{x}=1$