1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 02 (Class XI, Semester - 2)
Module Name/Title	Limits – Part 1
Module Id	Kemh_21301
Pre-requisites	Knowledge about Functions, Domain and Range of functions
Objectives	After going through this lesson, the learners will be able to understand the following: 1. Concept, notation, geometrical interpretation of a limit 2. Evaluation of left hand limit (LHL) and Right hand limit (RHL) 3. Algebra of limits 4. Limits of polynomials and rational functions
Keywords	Limits, LHL, RHL, Algebra of limits

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Introduction

Branch of mathematics which mainly deals with study of change in the value of a function as the point in the domain changes. This module is an introduction to 'calculus' which is a branch of mathematics. It deals with the concepts of limits of a functions, which is fundamental in understanding the concepts of derivatives.

Facts from history

- In the history of mathematics two names are prominent to share the credit for inventing calculus, Isaac Newton (1642-1727) and G.W. Leibnitz (1646-1717).

Limits

Suppose f is a function defined on an interval containing $x = a$ (Except possibly at $x = a$). Then 'informally, the limit is the real number L such that $f(x)$ is arbitrary close to L if x is chosen to be "sufficiently close to a" (excluding $x = a$)

Note: For a function to have a limit as $x \to a$, it is not necessary that the function be defined at the point $x = a$. When finding the limit we consider the values of the function in the neighborhood of the point a (very close to the point a) that are different from a

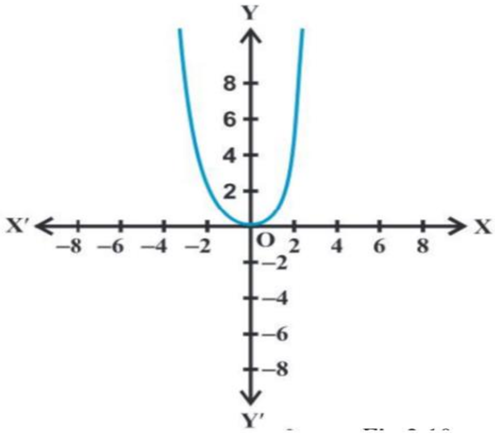
What does that really mean? Let see some functions and study their behavior graphically

Illustration 1

Consider the function $f(x) = x^2$. Observe that as x takes values very close to 0, the value of $f(x)$ also moves towards 0 . We say

$$\lim_{x \to 0} f(x) = 0$$

(to be read as limit of $f(x)$ as x tends to zero equals zero).



Facts from history

- The rigorous concept of calculus is mainly attributed to the great mathematicians, A.L. Cauchy, J.L. Lagrange and Karl Weierstrass. Cauchy gave the foundation of calculus. Cauchy used D' Alembert's limit concept to define the derivative of a function.

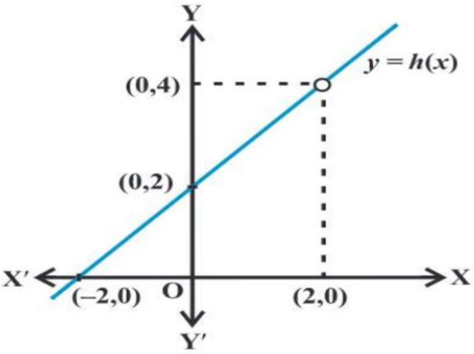
Also,

Illustration 2

Consider the following function.

$$h(x)=\frac{x^2-4}{x-2},\ x\neq 2.$$

Compute the value of $h(x)$ for values of x very near to 2 (but not at 2). Convince yourself that all these values are very near to 4.

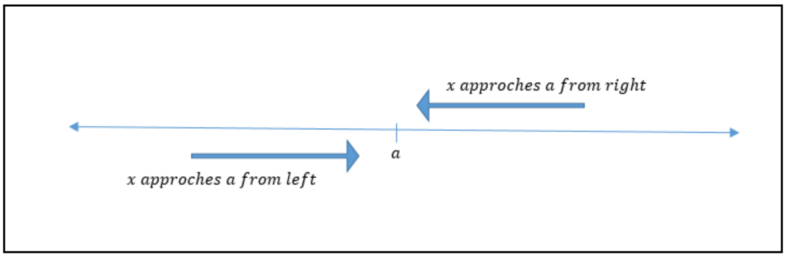


In both the cases above we can see that the limit of function as x approaches to some point is thought of as the value of the function should assume at that point, that is, the value of the function are assumed to be 0 and 4 respectively and is not attaining it as we are moving x closer and closer to the points 0 and 2

As you can see in this table for illustration 2

x	1.4	1.5	1.99	2	2.01	2.05	2.2
$f(x)$	3.4	3.5	3.99	Not defined	4.01	4.05	4.2

It is interesting to note that in both the illustrations the value which the function should assume at a given point $x = a$ did not really depend on how x is tending to a.



So, we can say!

Evaluation of Left Hand Limit (LHL) and Right Hand Limit (RHL)

There are essentially two ways x could approach a number a either from left or right, that is, all the values of x near to a could be less than a or greater than a but very close to a .This naturally leads to two limits

- ❖ The left hand limit
- ❖ The right hand limit

And if these two limits coincides then the common value is known as the limit of the function.

Left hand limit (LHL): The expected value of the function as dictated by the points to the left of a point defines the left hand limit of the function at that point which we can find by given steps in box

Steps (Left hand limit)

- ❖ Write $\lim_{x \to a^-} f(x)$
- ❖ Put $x = a - h$ and replace $x \to a^-$ by $h \to 0$
  To obtain $\lim_{h \to 0} f(a - h)$
- ❖ Simplify $\lim_{h \to 0} f(a - h)$
- ❖ The value obtained is the LHL of the function

Example: Find the left hand limit of the function where $f(x) = x(x + 1)$ as x tends to 5

Solution: LHL

(I) $\lim_{x \to 5^-} f(x)$

(II) $\lim_{x \to 5^-} f(x) = \lim_{h \to 0} f(5 - h)$

(III) $\lim_{h \to 0} f(5 - h) = \lim_{h \to 0}(5 - h)(5 - h + 1) = \lim_{h \to 0}(h^2 - 11h + 30)$

(IV) On simplification we get, $\lim_{x \to 5^-} f(x) = 30$

Similarly,

Right hand limit (RHL): The expected value of the function as dictated by the points to the right of a point defines the right hand limit of the function at that point which we can find by given steps in box

Steps (Right hand limit)

- ❖ Write $\lim_{x \to a^+} f(x)$
- ❖ Put $x = a + h$ and replace $x \to a^+$ by $h \to 0$
  To obtain $\lim_{h \to 0} f(a + h)$
- ❖ Simplify $\lim_{h \to 0} f(a + h)$
- ❖ The value obtained is the RHL of the function

Example: Find the right hand limit of the function where $f(x) = x(x + 1)$ as x tends to 5

Solution: RHL

(I) $\lim_{x \to 5^+} f(x)$

(II) $\lim_{x \to 5^+} f(x) = \lim_{h \to 0} f(5+h)$

(III) $\lim_{h \to 0} f(5+h) = \lim_{h \to 0}(5+h)(5+h+1) = \lim_{h \to 0}(h^2 + 11h + 30)$

(IV) On simplification we get, $\lim_{x \to 5^+} f(x) = 30$

Hence, as the LHL and RHL coincide. We say that the limit of the function exist and $\lim_{x \to 5} f(x) = 30$

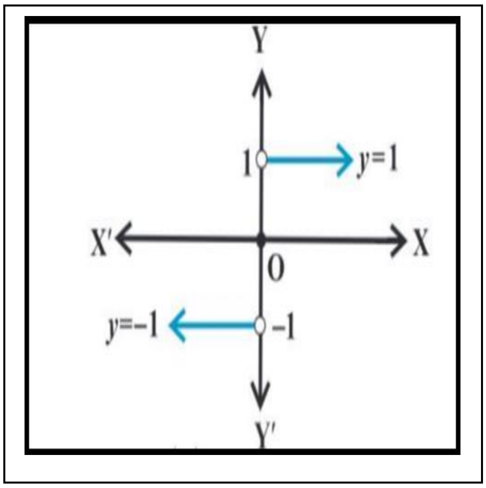
Therefore, in totality

We say $\lim_{x \to a^-} f(x)$ is the expected value of f at $x = a$ given the values of f near x to the left of a. This value is called the *left hand limit* of f at a.

We say $\lim_{x \to a^+} f(x)$ is the expected value of f at $x = a$ given the values of f near x to the right of a. This value is called the *right hand limit* of $f(x)$ at a.

If the right and left hand limits coincide, we call that common value as the limit of $f(x)$ at $x = a$ and denote it by $\lim_{x \to a} f(x)$.

Note: In case left hand limit and right hand limit are different. We say limit does not exist.



Example: Evaluate $\lim_{x \to 0} f(x)$ where $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$

Solution: RHL: $\lim_{x \to 0^+} f(x) = 1$

LHL: $\lim_{x \to 0^-} f(x) = -1$

But LHL≠RHL. Therefore, limit of the function as x tends to zero does not exist (even though the function is defined at 0)

Let us take some more examples and see what we observe

Example: Find the left hand limit and right hand limit of function $f(x) = 5x$ when $x \to 2$. Check whether $\lim_{x \to 2} f(x)$ exists.

Solution: $\lim_{x \to 2} f(x)$, where $f(x) = 5x$

LHL: $\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} 5(2-h) = 10$

RHL: $\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} 5(2+h) = 10$

As LHL =RHL =10

Therefore, $\lim_{x \to 2} 5x = 10$

Example: Evaluate $\lim_{x \to -1} f(x)$ where $f(x) = \begin{cases} x+5 & x \neq -1 \\ 0 & x = -1 \end{cases}$

Solution: we have to find $\lim_{x \to -1} f(x)$

LHL: $\lim_{x\to -1^-} f(x) = \lim_{h\to 0} f(-1-h) = \lim_{h\to 0}(-1-h)+5 = 4$

RHL: $\lim_{x\to -1^+} f(x) = \lim_{h\to 0} f(-1+h) = \lim_{h\to 0}(-1+h)+5 = 4$

As LHL =RHL =4

Therefore, $\lim_{x\to -1} f(x) = 4$

From the above examples we can make following observations between the value of a function at a point and the limit at that point, i.e., if $f(x)$ be a function and a be a point then we have following possibilities

- $\lim_{x\to a} f(x)$ exists but $f(a)$ does not exist

  $(f(a)\ \textit{means the value of the function}\ f(x)\ \textit{at point}\ a)$
- The value $f(a)$ exists but $\lim_{x\to a} f(x)$ does not exist
- $\lim_{x\to a} f(x)$ and $f(a)$ both exist but may or may not be equal

Algebra of limits

Also, notice that the functions taken are sum, product or subtraction of one or more functions. Such as, $\lim_{x\to 2} 5x$ where we may see $5x$ is a product of two functions 5 and x. Observed that the limiting process respects addition, subtraction, multiplication and division. Further, we can formalize these as theorems as below:

Theorem Let f and g be two functions such that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

(i) Limit of sum of two functions is sum of the limits of the functions, i.e.,

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

(ii) Limit of difference of two functions is difference of the limits of the functions, i.e.,

$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x).$$

(iii) Limit of product of two functions is product of the limits of the functions, i.e.,

$$\lim_{x\to a}[f(x)\,.\,g(x)] = \lim_{x\to a} f(x).\ \lim_{x\to a} g(x).$$

(iv) Limit of quotient of two functions is quotient of the limits of the functions (whenever the denominator is non zero), i.e.,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

Let us take some examples on it.

Example: Find the limit of the function $f(x) = x^2$ when x tends to 1.

Solution: We can see that the function $f(x) = x^2$ is product of the same function x.

Therefore, from above theorem, we say

$\lim_{x\to 1} f(x) = \lim_{x\to 1} x^2 = \lim_{x\to 1} x \,.\lim_{x\to 1} x$

$\lim_{x\to 1} x^2 = \mathbf{1}$

For example: Find the limit of the function $f(x) = (x^2 + 1)$ when x tends to 1.

Solution: We want to find $\lim_{x\to 1} f(x)$ where $f(x) = (x^2 + 1)$. We can see the above function is the sum of two functions, that is, $x^2 \; and \; 1$.

Therefore, from above theorem,

$\lim_{x\to 1}(x^2 + 1) = \lim_{x\to 1} x^2 + \lim_{x\to 1} 1$

$\lim_{x\to 1}(x^2 + 1) = 1 + 1 = \mathbf{2}$ $\quad as \; (\lim_{x\to a} k = k \; where \; a \; is \; any \; real \; number)$

Example: Find the limit of the function $f(x) = (x^3 - x^2)$ when x tends to π.

Solution: We want to find $\lim_{x\to \pi} f(x)$ where $f(x) = (x^3 - x^2)$. We can see the above function is difference between two functions, that is, $x^3 \; and \; x^2$.

Therefore, from above theorem, we can say

$\lim_{x\to \pi}(x^3 - x^2) = = \lim_{x\to \pi} x^3 - \lim_{x\to \pi} x^2$

$\lim_{x\to \pi}(x^3 - x^2) = (\pi^3 - \pi^2) = \boldsymbol{\pi^2(\pi - 1)}$.

Example: Find the limit of the function $f(x) = \frac{x^2+1}{x} \; where \; x \neq 0$ when x tends to 2.

Solution: We want to find $\lim_{x\to 2} f(x)$ where $f(x) = \frac{x^2+1}{x}$. We can see the above function is Quotient of two functions, that is, $x^2 + 1 \; and \; x$.

Therefore, from above theorem, we can say

$\lim_{x\to 2} \frac{x^2+1}{x} = \frac{\lim_{x\to 2} x^2+1}{\lim_{x\to 2} x} = \mathbf{\frac{5}{2}}$

Observation: We already know that the limit of a function can be found at a point where the function is not defined but observe if the point to which the variable tends to is a point in the domain of the function, then the value of the function at that point is its limit.

Limits of Polynomial and Rational Functions

Let us now see the limits of special types of functions:

Polynomial Function

A function f is said to be a polynomial function of degree n $f(x) = a_0 + a_1x + a_2x^2 + . . . + a_nx^n$, where a_is are real numbers such that $a_n \neq 0$ for some natural number n.

It is very interesting to note that if $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$ is polynomial function then $\lim_{x\to a} f(x) = f(a)$.

Justification is quite direst from the algebra of limits or you can see in web links of NCERT book at page 293

Example: Evaluate

I. $\lim_{x\to -1}(2 + 2x + 2x^2 - 5x^3)$

II. $\lim_{x\to 1}\left(x(1 + 2x - 3x^2)\right) + \lim_{x\to 1} e^x$

Solution: We know that when $f(x)$ a polynomial function then, $\lim_{x\to a} f(x) = f(a)$

(I) $\lim_{x\to -1}(2 + 2x + 2x^2 - 5x^3) = 2 + 2(-1) + 2(-1)^2 - 5(-1)^3 = 2 - 2 + 2 + 5$

$\lim_{x\to -1}(2 + 2x + 2x^2 - 5x^3) = 7$

(II) $\lim_{x\to 1}\left(x(1 + 2x - 3x^2)\right) + \lim_{x\to 1} e^x = 1(1 + 2 - 3) + e^1 = e$

Rational Function

A function f is said to be a rational function, if $f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials such that $h(x) \neq 0$. Then

$$\lim_{x\to a} f(x) = \lim_{x\to a}\frac{g(x)}{h(x)} = \frac{\lim_{x\to a} g(x)}{\lim_{x\to a} h(x)} = \frac{g(a)}{h(a)}$$

However, if $h(a)$ =0, then here can be two possibilities

- When $g(a)$ ≠0 and
- When $g(a)$ =0

In the former case we say limit does not exist whereas

In the latter case where $\lim_{x\to a}\frac{f(x)}{g(x)}$ takes the form $\frac{0}{0}$ the limit may or may not be defined. The determination of limit in such a case is traditionally referred to as the evaluation of the indeterminate form $\frac{0}{0}$ (undetermined or illusionary form). In such case we try to rewrite the function cancelling the factors

Facts:

There are in total 7 indeterminate forms, namely,

- $\frac{0}{0}$
- $\frac{\infty}{\infty}$
- $0 \times \infty$
- $\infty - \infty$
- 0^0
- ∞^0
- 1^∞

which are causing the limit to be of the form $\frac{0}{0}$ (either by factorization, rationalization method or many more).

Example: Find the limits:

(I) $\lim\limits_{x \to 2} \frac{x^2-4}{x^3-4x^2+4x}$

(II) $\lim\limits_{x \to 1} \frac{x-2}{x^2-x} - \frac{1}{x^3-3x^2+2x}$

Solution:

(I) Evaluating the function at 2, we get it of the form $\frac{0}{0}$

Hence, $\lim\limits_{x \to 2} \frac{x^2-4}{x^3-4x^2+4x} = \lim\limits_{x \to 2} \frac{(x+2)(x-2)}{x(x-2)^2}$

$$\lim_{x \to 2} \frac{x^2-4}{x^3-4x^2+4x} = \frac{2+2}{2(2-2)} = \frac{4}{0}$$

Which is not defined.

(II) Evaluating the function at 1, we get it of the form $\frac{0}{0}$

Hence, $\lim\limits_{x \to 1} \frac{x-2}{x^2-x} - \frac{1}{x^3-3x^2+2x} = \lim\limits_{x \to 1} \frac{x^2-4x+3}{x(x-1)(x-2)}$

$$\lim_{x \to 1} \frac{x-2}{x^2-x} - \frac{1}{x^3-3x^2+2x} = \lim_{x \to 1} \frac{(x-3)(x-1)}{x(x-1)(x-2)} = \lim_{x \to 1} \frac{(x-3)}{x(x-2)}$$

$$\lim_{x \to 1} \frac{x-2}{x^2-x} - \frac{1}{x^3-3x^2+2x} = \frac{1-3}{1(1-2)} = 2$$

Example: Evaluate $\lim\limits_{x \to 1} \frac{x^4-1}{x-1}$

Solution: When we put $x = 0$, the expression takes the $\frac{0}{0}$ form.

Therefore we will try to rewrite the function cancelling the factors which are causing the limit to be of the form $\frac{0}{0}$

We know that $x^4 - 1$ can be written as,

$x^4 - 1 = (x-1)(x^3 + x^2 + x + 1)$

Thus, $\lim\limits_{x \to 1} \frac{x^4-1}{x-1} = \lim\limits_{x \to 1} (x^3 + x^2 + x + 1)$

$$\lim_{x \to 1} \frac{x^4-1}{x-1} = 1 + 1 + 1 + 1 = 4$$

Hence, $\lim\limits_{x \to 1} \frac{x^4-1}{x-1} = 4$

Observe the above example carefully! Yes, we may notice an important limit as given below:

Theorem For any positive integer n,

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}.$$

Remark The expression in the above theorem for the limit is true even if n is any rational number and a is positive.

Proof Dividing $(x^n - a^n)$ by $(x - a)$, we see that

$$x^n - a^n = (x-a)\,(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + x\,a^{n-2} + a^{n-1})$$

$$\text{Thus, } \lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} (x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + x\,a^{n-2} + a^{n-1})$$

$$= a^{n-1} + a\,a^{n-2} + \ldots + a^{n-2}(a) + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + \ldots + a^{n-1} + a^{n-1} \ (n \text{ terms})$$

$$= na^{n-1}$$

Example: Evaluate the following

I. $\displaystyle \lim_{x \to 2} \frac{\frac{1}{x} + \frac{1}{2}}{x+2}$

II. $\displaystyle \lim_{x \to 5} \frac{x\sqrt{x} - 5\sqrt{5}}{x-5}$

Solution:

Part (I)

Solution: When we put $x = 2$ the expression takes the form of $\frac{0}{0}$

$$\text{Now, } \lim_{x \to 2} \frac{\frac{1}{x} + \frac{1}{2}}{x+2} = \lim_{x \to 2} \frac{x+2}{(x+2)2x}$$

$$= \lim_{x \to 2} \frac{1}{2x}$$

$$= \mathbf{\frac{1}{4}}$$

Part (II)

When we put $x = 5$ the expression takes the form of $\frac{0}{0}$

$$\text{Now, } \lim_{x \to 5} \frac{x\sqrt{x} - 5\sqrt{5}}{x-5} = \lim_{x \to 5} \frac{x^{3/2} - 5^{3/2}}{x-5}$$

$$= \frac{3}{2}(5)^{1/2}$$

$$= \mathbf{\frac{3}{2}\sqrt{5}}$$

Example: Find the value of k, if $\lim\limits_{x \to 1} \frac{x^4-1}{x-1} = \lim\limits_{x \to k} \frac{x^3-k^3}{x^2-k^2}$

Solution:

We have, $\lim\limits_{x \to 1} \frac{x^4-1}{x-1} = 4$……….(1)

and $\lim\limits_{x \to k} \frac{x^3-k^3}{x-k} \times \frac{x-k}{x^2-k^2}$

$= \lim\limits_{x \to k} \frac{x^3-k^3}{x-k} \div \lim\limits_{x \to k} \frac{x^2-k^2}{x-k}$ (By algebra of limits)

$= 3k^{3-1} \div 2k^{2-1} = \frac{3}{2}k$ ………(2)

Therefore, from (1) and (2), we get

$$4 = \frac{3k}{2}$$

$$k = \frac{8}{3}$$

Hence, $k = \frac{8}{3}$

Problems

Let us have some mixed problems on what we have studied above in this module.

Question: Evaluate $\lim\limits_{x \to 0} \frac{\sqrt{1+x}-1}{x}$

Solution: When we put $x = 1$ the expression takes the form of $\frac{0}{0}$

Now, $\lim\limits_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \lim\limits_{y \to 1} \frac{\sqrt{y}-1}{y-1}$

$$= \lim_{y \to 1} \frac{y^{1/2}-1}{y-1}$$

$$= \frac{1}{2}(1)^{\frac{1}{2}-1} = \mathbf{\frac{1}{2}}$$

Question: Evaluate $\lim\limits_{x \to 2} \left(\frac{2}{4-x^2} + \frac{1}{x-2}\right)$.

Solution: we have,

$$= \lim_{x \to 2} \left(\frac{4}{4-x^2} + \frac{1}{x-2}\right)$$

$$= \lim_{x \to 2} \left(\frac{4-(2+x)}{4-x^2}\right)$$

$$= \lim_{x \to 2} \left(\frac{2+x}{4-x^2}\right)$$

$= \lim\limits_{x \to 2} \left(\frac{1}{2+x}\right) = \mathbf{\frac{1}{4}}$

Question: Evaluate $\lim\limits_{x\to\sqrt{2}} \frac{x^4-4}{x^2+3x\sqrt{2}-8}$

Solution: When $x=\sqrt{2}$, the expression assumes the form $\frac{0}{0}$

On factorizing the numerator and denominator, we get

$$\lim_{x\to\sqrt{2}} \frac{x^4-4}{x^2+3x\sqrt{2}-8}$$

$$= \lim_{x\to\sqrt{2}} \frac{(x^2-2)(x^2+2)}{(x+4\sqrt{2})(x-\sqrt{2})}$$

$$= \lim_{x\to\sqrt{2}} \frac{(x-\sqrt{2})(x+\sqrt{2})(x^2+2)}{(x+4\sqrt{2})(x-\sqrt{2})}$$

$$= \lim_{x\to\sqrt{2}} \frac{(x+\sqrt{2})(x^2+2)}{(x+4\sqrt{2})} = \frac{(2\sqrt{2})(2+2)}{5\sqrt{2}} = \mathbf{\frac{8}{5}}$$

Question: Evaluate $\lim\limits_{x\to 0} \frac{\sqrt{9+x^2}-\sqrt{9-x^2}}{x^2}$

Solution: We can see that if we put $x=0$, the expression takes the $\frac{0}{0}$ form.

Therefore, on rationalizing the numerator, we get

$$\lim_{x\to 0} \frac{\sqrt{9+x^2}-\sqrt{9-x^2}}{x^2} = \lim_{x\to 0} \frac{(\sqrt{9+x^2}-\sqrt{9-x^2})(\sqrt{9+x^2}+\sqrt{9-x^2})}{x^2\,(\sqrt{9+x^2}+\sqrt{9-x^2})}$$

$$= \lim_{x\to 0} \frac{9+x^2-9+x^2}{x^2\,(\sqrt{9+x^2}+\sqrt{9-x^2})} = \frac{2}{\sqrt{9}+\sqrt{9}} = \mathbf{\frac{1}{3}}$$

Question: Evaluate $\lim\limits_{x\to 4} \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}$.

Solution: Rationalizing the numerator and denominator, we get

$$\lim_{x\to 4} \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}} = \lim_{x\to 4} \frac{(3-\sqrt{5+x})(3+\sqrt{5+x})(1+\sqrt{5-x})}{(1-\sqrt{5-x})(3+\sqrt{5+x})(1+\sqrt{5-x})}$$

$$\lim_{x\to 4} \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}} = \lim_{x\to 4} \frac{(9-5-x)(1+\sqrt{5-x})}{(1-5+x)(3+\sqrt{5+x})}$$

$$\lim_{x\to 4} \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}} = \lim_{x\to 4} \frac{-(x-4)(1+\sqrt{5-x})}{(x-4)(3+\sqrt{5+x})}$$

$$\lim_{x\to 4} \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}} = \lim_{x\to 4} \frac{-(1+\sqrt{5-x})}{(3+\sqrt{5+x})} = -\mathbf{\frac{1}{3}}$$

Question: If $f(x) = \begin{cases} mx^2+n & x<0 \\ nx+m & 0\le x\le 1 \\ nx^3+m & x>1 \end{cases}$. For what integers m and n does both $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 1} f(x)$ exist?

Solution: It is given that $\lim\limits_{x\to 0} f(x)$ exist. This implies LHL=RHL

(LHL of $f(x)\ at\ x = 0$)

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} m(0-h)^2 + n = n$$

(RHL of $f(x)\ at\ x = 0$)

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} n(0+h) + m = m$$

Therefore, $\boldsymbol{\lim_{x\to 0} f(x)}$ exist if $n = m$

Similarly, it is also given that $\boldsymbol{\lim_{x\to 1} f(x)}$ exist. This implies LHL=RHL

(LHL of $f(x)\ at\ x = 1$)

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} n(1+h) + m = n + m$$

(RHL of $f(x)\ at\ x = 1$)

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} n(1+h)^3 + m = n + m$$

Therefore, $\boldsymbol{\lim_{x\to 1} f(x)}$ exist for any value of $m\ and\ n$

Hence, $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ both exist for $n = m$

Before ending, let us recall what we have studied in this module

Summary

- **Left hand limit (LHL):** The expected value of the function as dictated by the points to the left of a point defines the left hand limit of the function at that point.
- **Right hand limit (RHL):** The expected value of the function as dictated by the points to the right of a point defines the right hand limit of the function at that point.
- **Limit:** Limit of a function at a point is the common value of the left and right hand limits, if they coincide.
- **Notation:**
  - Limit: $\lim_{x\to a} f(x)$
  - LHL: $\lim_{x\to a^-} f(x)$
  - RHL: $\lim_{x\to a^+} f(x)$
- **Algebra of Limits:** For functions $f\ and\ g$ the following holds:
  - $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
  - $\lim_{x\to a}(f(x).g(x)) = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$
  - $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$
- **Some standard limits**
  - $\lim_{x\to a} k = k$ $\quad$ $where\ k\ is\ any\ real\ fixed\ number$
  - $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$