1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 02 (Class XI, Semester - 2)
Module Name/Title	Introduction to Three Dimensional Geometry-Part 2
Module Id	kemh_21202
Pre-requisites	Basic knowledge of Three Dimensional Geometry
Objectives	After going through this lesson, the learners will be able to do the following: • Section Formula (Internal Division) • Section Formula (External Division) • Mid-Point Formula • Centroid of a Triangle
Keywords	Internal Division, External Division, Mid Point Formula, Centroid of a Triangle

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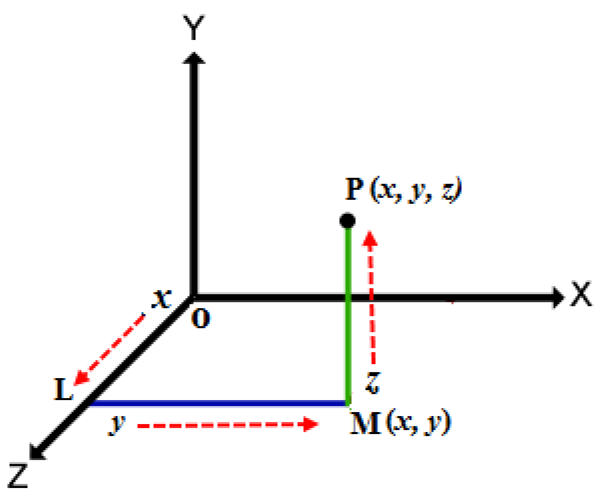
1. Introduction:

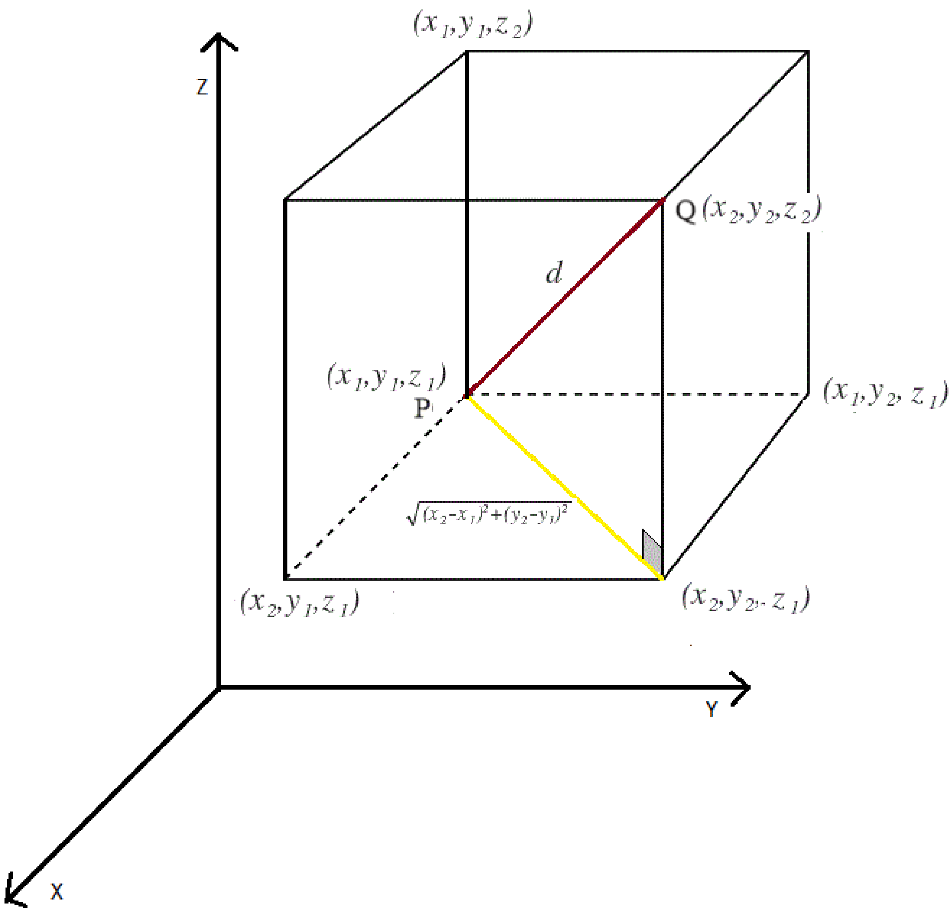
In previous module, you have learnt to locate a point in three dimensional space using a triplet (x, y, z), the coordinates of a point P in the space. You have noticed that, we have an additional coordinate 'z' , in triplet (x, y, z), compared to pair (x, y) used as coordinates of a point, due to extension from two dimensional plane to three dimensional space.



You have also learnt distance formula in three dimensional geometry.

To find distance between two points P(x_1, y_1, z_1) and Q (x_2, y_2, z_2), we use the formula,

$$PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$



Here, also we have an additional term, $(z_2 - z_1)^2$.

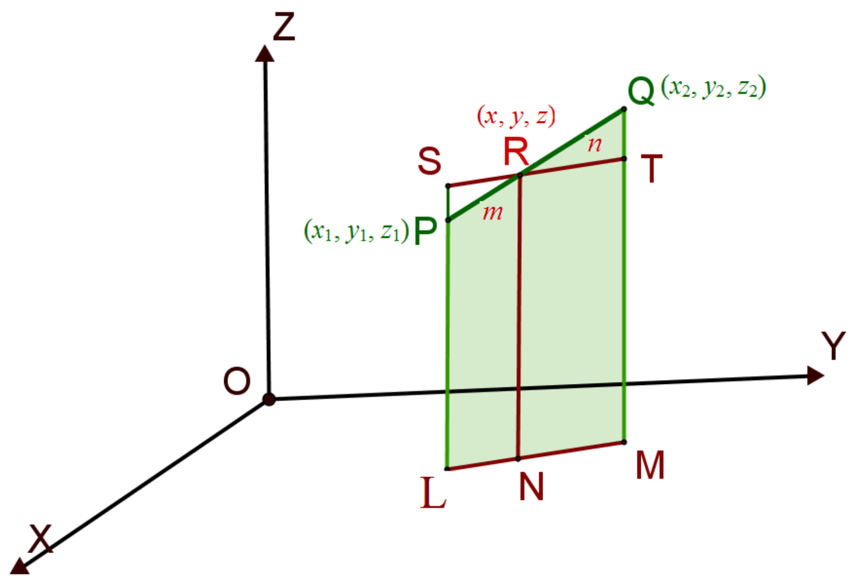
Let us now extend **Section formula** learnt in two dimensional geometry in earlier classes to three dimensional space.

The section formula gives us the coordinates of a point which divides a given line segment in a given ratio.

2. **Section Formula (Internal division):**

Let us consider two points P(x_1, y_1, z_1) and Q (x_2, y_2, z_2), and the point R (x, y, z) which divides the line segment PQ in the ratio $m : n$ internally. Internal division means point R lies in between points P and Q, on line segment PQ.

Draw PL, QM and RN perpendicular to the XY-plane. Obviously PL || RN || QM and feet of these perpendiculars will lie in a XY-plane.



Also the points L, M and N will lie on a line which is the intersection of the plane containing PL, RN and QM with the XY-plane. Through the point R a line ST is drawn parallel to the line LM. The line ST is intersecting the line segment LP externally at point S and internally the line segment

MQ at T, as is shown in figure above. Since, PL || RN || QM, therefore, quadrilaterals LNRS and NMTR are parallelograms and the triangles PSR and QTR are similar.

Hence,

$$\frac{m}{n} = \frac{\mathrm{PR}}{\mathrm{QR}} = \frac{\mathrm{SP}}{\mathrm{QT}}$$

$$= \frac{\mathrm{SL} - \mathrm{PL}}{\mathrm{QM} - \mathrm{TM}}$$

$$= \frac{\mathrm{NR} - \mathrm{PL}}{\mathrm{QM} - \mathrm{NR}}$$

Therefore, we get,

$$\frac{m}{n} = \frac{z - z_1}{z_2 - z}$$

Cross-multiplying we get,

$$z = \frac{mz_2 + nz_1}{m+n}$$

Similarly, by drawing perpendiculars to the XZ and YZ-planes, we get

$$y = \frac{my_2 + ny_1}{m + n} \quad \text{and} \quad x = \frac{mx_2 + nx_1}{m + n}$$

Hence, the coordinates of the point R which divides the line segment joining points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio $m : n$ are,

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

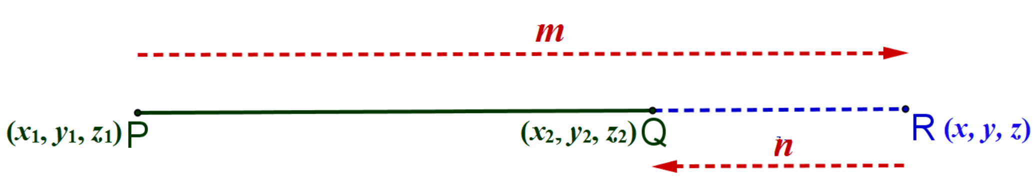
Remark:

The coordinates of the point R which divides the line segment PQ in the ratio k : 1 can be obtained by taking $k = \frac{m}{n}$, Hence, the required coordinates of point R are,

$$\left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}, \frac{kz_2+z_1}{k+1}\right)$$

3. Section Formula (External division):

Let us consider two points P(x_1, y_1, z_1) and Q (x_2, y_2, z_2), and the point R (x, y, z) dividing the line segment PQ in the ratio $m : n$ externally, then the point R will be on PQ produced, such that PR : RQ = $m : n$, see the figure below the line segments PR and RQ have opposite directions, hence, coordinates of point R can be obtained by replacing n by $-n$,



So, coordinates of point R are;

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$$

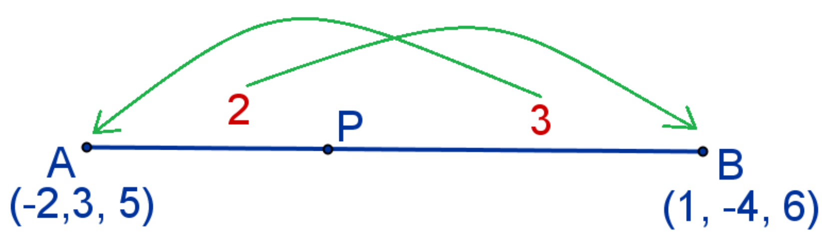
Example:

Find coordinates of the point which divides the line segment joining the points (–2, 3, 5) and (1, – 4, 6) in the ratio,

(i) 2 : 3 internally (ii) 2 : 3 externally.

Solution:

(i) Let P (x, y, z) be the point which divides the line segment joining points A (–2, 3, 5) and B (1, – 4, 6) internally in the ratio 2 : 3.



Then, using section formula,

$$x = \frac{2(1)+3(-2)}{2+3} = \frac{2-6}{5} = \frac{-4}{5}$$

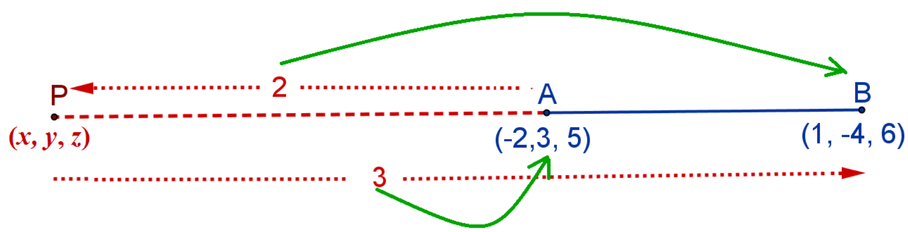
$$y = \frac{2(-4)+3(3)}{2+3} = \frac{-8+9}{5} = \frac{1}{5}$$

$$z = \frac{2(6)+3(5)}{2+3} = \frac{12+15}{5} = \frac{27}{5}$$

Thus, the coordinates of the required point P are;

$$\left(\frac{-4}{5}, \frac{1}{5}, \frac{27}{5}\right).$$

(ii) In the figure below, the point P (x, y, z) divides the line segment joining points A (–2, 3, 5) and B (1, – 4, 6) externally in the ratio 2 : 3,



according to the rule for external division explained above, the coordinates of point P are;

$$x = \frac{2(1)-3(-2)}{2-3} = \frac{2+6}{-1} = -8$$

$$y = \frac{2(-4)-3(3)}{2-3} = \frac{-8-9}{-1} = 17$$

$$z = \frac{2(6)-3(5)}{2-3} = \frac{12-15}{-1} = 3$$

Hence, the coordinates of the required point P dividing the line segment AB externally in the ratio 2 : 3 are; (– 8, 17, 3).

Example:

A point R with *x*-coordinate equal to 4 lies on the line segment joining the points P (2, –3, 4) and Q (8, 0, 10). Find *y*-coordinate and *z*-coordinate of the point R.

Solution:

Let us suppose point R divides the line segment PQ in the ratio k : 1. Then *x*, *y* and *z* coordinates of the point R are given by

$$x = \frac{8k+2}{k+1}, \quad y = \frac{-3}{k+1}, \quad z = \frac{10k+4}{k+1}$$

Since *x*-coordinate of the point R is 4, therefore,

$$4 = \frac{8k+2}{k+1}$$

Solving, we get, $k = \frac{1}{2}$, substituting the value of k above we get,

$$y = -2 \quad \text{and} \quad z = 6,$$

Hence, y-coordinate of the point R is – 2 and z-coordinate of the point R is 6.

4. Mid-point formula:

Let us take two points P(x_1, y_1, z_1) and Q (x_2, y_2, z_2) and M (x, y, z) be the mid-point of the line segment PQ, then PM = MQ, hence, the point M divides the line segment PQ in the ratio 1:1 internally, Putting, $m = n = 1$, in the formula for internal division, the coordinates of mid-point of PQ are given by,

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2} \quad \text{and} \quad z = \frac{z_1 + z_2}{2}$$

Hence, coordinates of point M are,

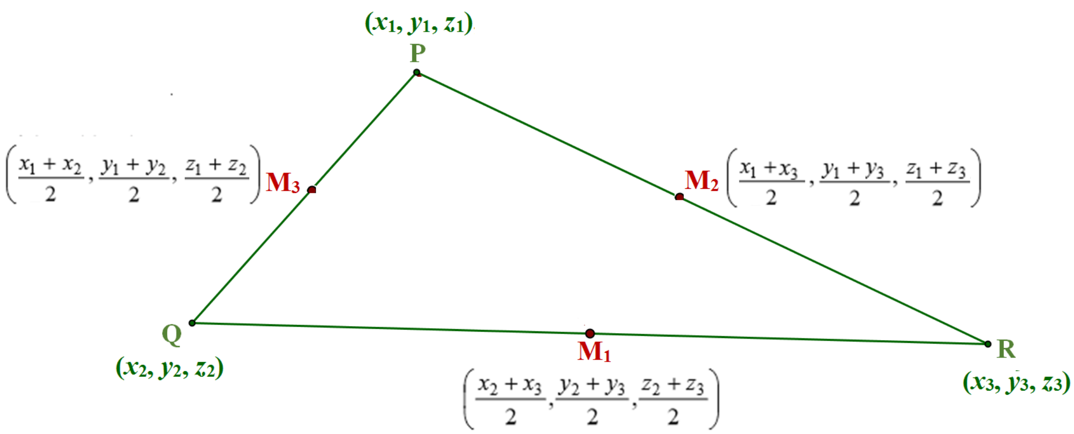
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

Example:

The mid-points of the sides of a triangle are (1, 5, –1), (0, 4, –2) and (2, 3, 4). Find the vertices of the triangle.

Solution:

Let the vertices of the triangle be P (x_1, y_1, z_1), Q (x_2, y_2, z_2) and R (x_3, y_3, z_3) and let M_1 (1, 5, –1), M_2 (0, 4, –2) and M_3 (2, 3, 4) be the mid-points of the sides QR, RP and PQ.



Since M_1 is the mid-point of side QR, therefore,

$$\frac{x_2+x_3}{2} = 1, \qquad \frac{y_2+y_3}{2} = 5\ , \qquad \frac{z_2+z_3}{2} = -1$$

$$x_2 + x_3 = 2, \qquad y_2 + y_3 = 10, \qquad z_2 + z_3 = -2 \ldots\ldots\ldots\ldots \text{(i)}$$

M_2 is the mid-point of side PR, therefore,

$$\frac{x_1+x_3}{2} = 0, \qquad \frac{y_1+y_3}{2} = 4\ , \qquad \frac{z_1+z_3}{2} = -2$$

$$x_1 + x_3 = 0, \qquad y_1 + y_3 = 8, \qquad z_1 + z_3 = -4 \ldots\ldots\ldots\ldots \text{(ii)}$$

M_3 is the mid-point of side PQ, therefore,

$$\frac{x_1+x_2}{2} = 2, \qquad \frac{y_1+y_2}{2} = 3\ , \qquad \frac{z_1+z_2}{2} = 4$$

$$x_1 + x_2 = 4, \qquad y_1 + y_2 = 6, \qquad z_1 + z_2 = 8 \ldots\ldots\ldots\ldots \text{(iii)}$$

Adding the three equations,

$$2(x_1 + x_2 + x_3) = 6, \quad 2(y_1 + y_2 + y_3) = 24, \quad 2(z_1 + z_2 + z_3) = 2$$

Or, $(x_1 + x_2 + x_3) = 3$, $(y_1 + y_2 + y_3) = 12$, $(z_1 + z_2 + z_3) = 1$

…………(iv)

Using equation (iv) with equations (i), (ii) and (iii), we get,

$x_1 = 1, \quad x_2 = 3, \quad x_3 = -1$

$y_1 = 2, \quad y_2 = 4, \quad y_3 = 6$

$z_1 = 3, \quad z_2 = 5, \quad z_3 = -7$

Hence, coordinates of the vertices of the triangle are,

(1, 2, 3), (3, 4, 5) and (–1, 6, –7)

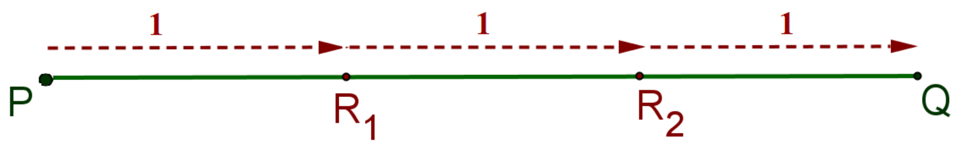
Example:

Find coordinates of the points which trisect the line segment joining

the points P (2, 1, –3) and Q (5, – 8, 3).

Solution:

Let R_1 and R_2 be the points which trisect the line segment PQ.



Then, $PR_1 = R_1R_2 = R_2Q$, hence, R_1 divides the line segment PQ in the ratio 1 : 2 and R_2 divides PQ in the ratio 2 : 1, Therefore, for point R_1,

$$x = \frac{1(5)+2(2)}{1+2} = \frac{5+4}{3} = 3$$

$$y = \frac{1(-8)+2(1)}{1+2} = \frac{-8+2}{3} = -2$$

$$z = \frac{1(3)+2(-3)}{1+2} = \frac{3-6}{3} = -1$$

Hence, coordinates of the point R_1 are (3, –2, –1).

And point R_2 divides PQ in the ratio 2 : 1, Therefore, for point R_2, we have,

$$x = \frac{2(5)+1(2)}{2+1} = \frac{10+2}{3} = 4$$

$$y = \frac{2(-8)+1(1)}{2+1} = \frac{-16+1}{3} = -5$$

$$z = \frac{2(3)+1(-3)}{2+1} = \frac{6-3}{3} = 1$$

Therefore, coordinates of the point R_2 are (4, –5, 1).

Coordinates of the points which trisect the given line segment are;

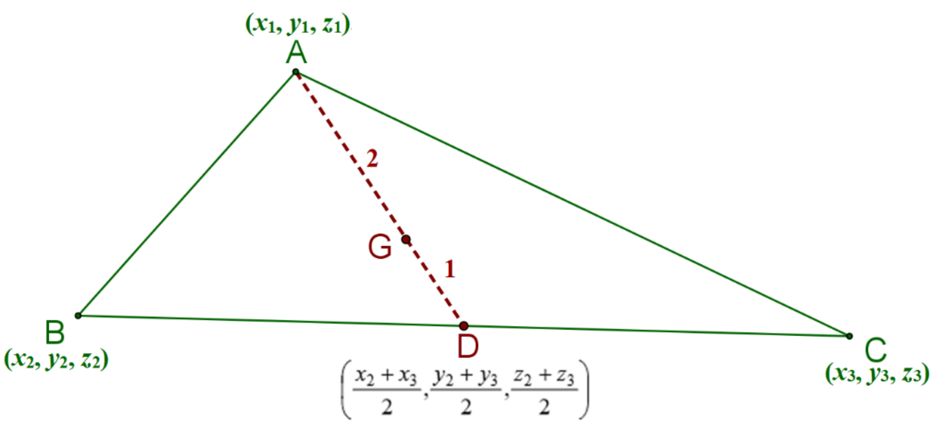
(3, –2, –1) and (4, –5, 1).

5. Centroid of a triangle:

Let us now find out the coordinates of centroid of a triangle whose vertices are known. Let ABC be the triangle and the coordinates of the vertices A, B and C be (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively. Let D be the mid-point of side BC of the triangle. Then the coordinates of point D will be,

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right)$$

Let G be the centroid of the triangle.



then G will divide the median AD in the ratio 2 : 1. Hence, the coordinates of G will be;

$$\left(\frac{2\left(\frac{x_2 + x_3}{2}\right) + x_1}{2+1}, \frac{2\left(\frac{y_2 + y_3}{2}\right) + y_1}{2+1}, \frac{2\left(\frac{z_2 + z_3}{2}\right) + z_1}{2+1} \right)$$

or $$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right)$$

6. Example:

If the origin is the centroid of the triangle PQR with vertices P (2*a*, 2, 6), Q (– 4, 3*b*, –10) and R (8, 14, 2*c*), then find the values of *a*, *b* and *c*.

Solution:

We know that coordinates of the centroid of a triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is,

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right),$$

The vertices of the given triangle are, P (2*a*, 2, 6), Q (– 4, 3*b*, –10) and R (8, 14, 2*c*), let centroid of the triangle PQR be G (*x*, *y*, *z*) then,

$$x = \frac{2a-4+8}{3} \quad y = \frac{2+3b+14}{3} \quad z = \frac{6-10+2c}{3}$$

It is given that the origin is the centroid of the triangle PQR, therefore,

$$0 = \frac{2a-4+8}{3}, \quad 0 = \frac{2+3b+14}{3}, \quad 0 = \frac{6-10+2c}{3}$$

Solving we get, $a = -2$, $b = -\frac{16}{3}$ and $c = 2$

Example:

The centroid of a triangle PQR is at the point (1, 1, 1). If the coordinates of the vertices P and Q are (3, –5, 7) and (–1, 7, – 6) respectively, find the coordinates of the vertex R.

Solution:

Let the coordinates of vertex R be (x, y, z) and the centroid of the triangle be G, the coordinates of centroid are given (1, 1, 1). then,

$$\frac{x+3-1}{3} = 1, \quad \frac{y-5+7}{3} = 1, \quad \frac{z+7-6}{3} = 1$$

Solving we get, $x = 1$, $y = 1$ and $z = 2$,

Hence, coordinates of vertex R are (1, 1, 2).

Example:

Find the ratio in which the line segment joining the points P (2, 3, 4) and Q (– 3, 5, – 4) is divided by the YZ-plane. Also find the coordinates of the point of intersection.

Solution:

Let YZ-plane divide the line segment joining points P (2, 3, 4) and Q (– 3, 5, – 4) at point R (x, y, z) in the ratio k : 1. Then the coordinates of point R are

$$\left(\frac{-3k+2}{k+1}, \frac{5k+3}{k+1}, \frac{-4k+4}{k+1}\right)$$

Since point R lies on YZ-plane, the x-coordinate of point R should be zero, therefore,

$$\frac{-3k+2}{k+1} = 0 \ ,$$

$$k = \frac{2}{3}$$

Hence, YZ-plane divides the join of points P (2, 3, 4) and Q (– 3, 5, – 4) in the ratio 2 : 3 internally.

Putting value of k , the coordinates of the required point are,

$$\left(\frac{-3\left(\frac{2}{3}\right)+2}{\left(\frac{2}{3}\right)+1}, \frac{5\left(\frac{2}{3}\right)+3}{\left(\frac{2}{3}\right)+1}, \frac{-4\left(\frac{2}{3}\right)+4}{\left(\frac{2}{3}\right)+1}\right)$$

$$= \left(0, \frac{19}{5}, \frac{4}{5}\right)$$

Thus the coordinates of the point R, the point of intersection of the line segment PQ and the YZ-plane is $\left(0, \frac{19}{5}, \frac{4}{5}\right)$

Example:

Use section formula to prove that the three points P (– 2, 3, 5), Q (1, 2, 3) and R (7, 0, –1) are collinear.

Solution:

Suppose the given points are collinear and the point R divides PQ in the ratio k : 1, then coordinates of point R will be given by,

$$\left(\frac{k-2}{k+1}, \frac{2k+3}{k+1}, \frac{3k+5}{k+1}\right)$$

But, coordinates of point R are (7, 0, –1), thus

$\frac{k-2}{k+1} = 7$, $\frac{2k+3}{k+1} = 0$ and $\frac{3k+5}{k+1} = -1$,

Solution of each of these three gives,

$$k = \frac{-3}{2}$$

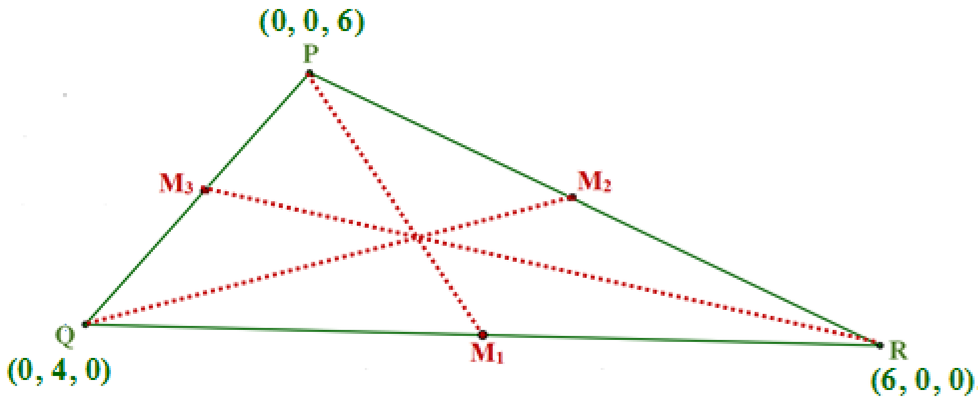
Since each of these three equations gives same value of k, hence, the given points are collinear and the point R divides PQ externally in the ratio 3 : 2.

Example:

Find the lengths of the median of the triangle with vertices P (0, 0, 6), Q (0, 4, 0) and R (6, 0, 0).

Solution:

Let M_1, M_2 and M_3 be the mid-points of the sides QR, RP and PQ, then



by mid-point formula coordinates of points M_1, M_2 and M_3 are respectively,

M_1 : $\left(\frac{0+6}{2}, \frac{4+0}{2}, \frac{0+0}{2}\right)$ or (3, 2, 0)

M_2 : $\left(\frac{6+0}{2}, \frac{0+0}{2}, \frac{0+6}{2}\right)$ or (3, 0, 3)

M_3 : $\left(\frac{0+0}{2}, \frac{0+4}{2}, \frac{6+0}{2}\right)$ or (0, 2, 3)

Using distance formula, length of the median PM_1 is;

$$PM_1 = \sqrt{(3-0)^2 + (2-0)^2 + (0-6)^2}$$

$$= \sqrt{9+4+36} = \sqrt{49} = 7$$

length of the median PM_2 is;

$$PM_2 = \sqrt{(3-0)^2 + (0-4)^2 + (3-0)^2}$$

$$= \sqrt{9+16+9} = \sqrt{34}$$

and length of the median PM_3 is;

$$PM_3 = \sqrt{(0-6)^2 + (2-0)^2 + (3-0)^2}$$

$$= \sqrt{36+4+9} = \sqrt{49} = 7$$

7. Summary:

1. The section formula gives us the coordinates of that point of a line segment which divides the given line segment in a given ratio.
2. The coordinates of the point R which divides the line segment joining points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio $m : n$ are,

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n} \right)$$

3. The coordinates of the point R which divides the line segment joining points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) externally in the ratio $m : n$ are,

$$\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}, \frac{mz_2 - nz_1}{m - n} \right)$$

4. The coordinates of the mid-point of the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

5. The coordinates of the centroid of the triangle, whose vertices are (x_1, y_1, z_1) (x_2, y_2, z_2) and (x_3, y_3, z_3), are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right)$$