1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 02 (Class XI, Semester - 2)
Module Name/Title	Introduction to Three Dimensional Geometry-Part 1
Module I	kemh_21201
Pre-requisites	Basic knowledge of Three Dimensional Geometry
Objectives	After going through this lesson, the learners will be able to do the following: • Coordinate Axes and Coordinate Planes in 3D Space • Coordinates of a Point in Space • Distance between two points
Keywords	Coordinate Axes, Coordinate Planes, Point in Space, Distance between two points

2. Development Team

Role	Name	Affiliation
National MOOC Coordinator (NMC)	Prof. Amarendra P. Behera	CIET, NCERT, New Delhi
Program Coordinator	Dr. Rejaul Karim Barbhuiya	CIET, NCERT, New Delhi
Course Coordinator (CC) / PI	Prof. Til Prasad Sarma	DESM, NCERT, New Delhi
Course Co-Coordinator /Co- PI	Ms. Anjali Khurana	CIET, NCERT, New Delhi
Subject Matter Expert (SME)	Dr. Sadhna Shrivastava (Retd.)	KVS, Faridabad, Haryana
Review Team	Prof. Bhim Prakash Sarrah	Assam University, Tezpur

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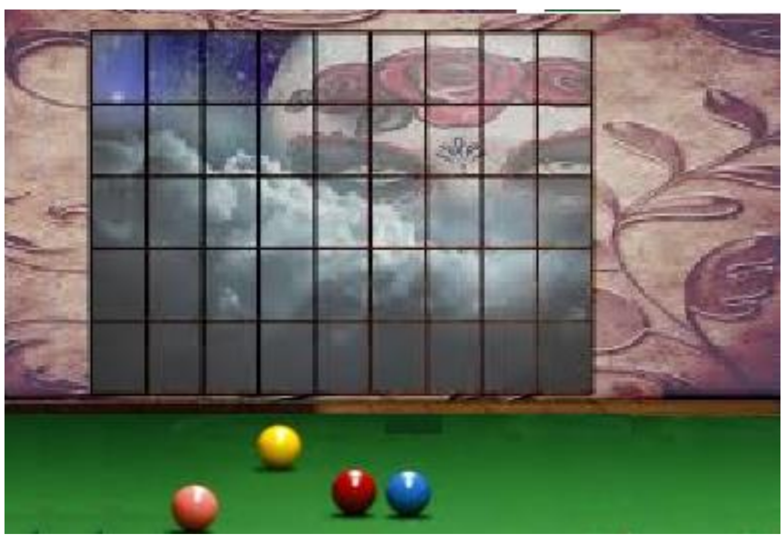
1. Introduction:

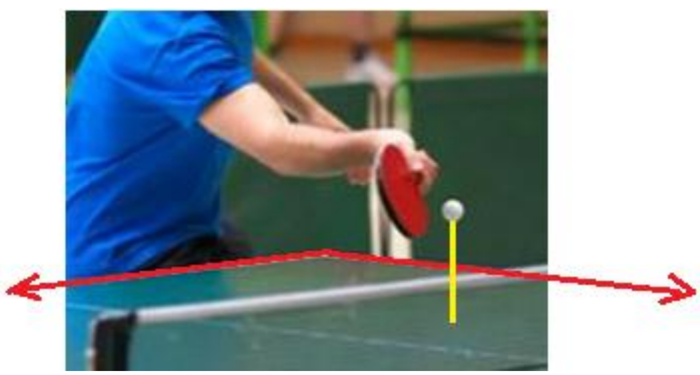
You have learnt two dimensional coordinate geometry in earlier classes and are familiar with locating a given point in two dimensional plane, with reference to two perpendicular lines, known as coordinate axes.



Suppose we have a room and some balls are placed on its floor, then their respective locations can be well specified using knowledge of two dimensional coordinate geometry. Similarly using two intersecting mutually perpendicular lines in the plane and with an ordered pair of numbers known as coordinates, we can specify location of any moving car on road, position of any specific ball on billiard table, at any time of

its motion and position of any specific student sitting in the class-room.

But in actual life, we do not deal with objects lying in a plane only. For example, consider different positions of the ball in table-tennis game



or location of shuttle cork at different points of time in the game badminton, it requires one more dimension to depict its height. Consider the following picture, there are many objects, all are lying in a plane i.e.,

on the ground, hence, their positions can be specified using two-dimensional coordinate geometry. But as soon as the plane takes off, we

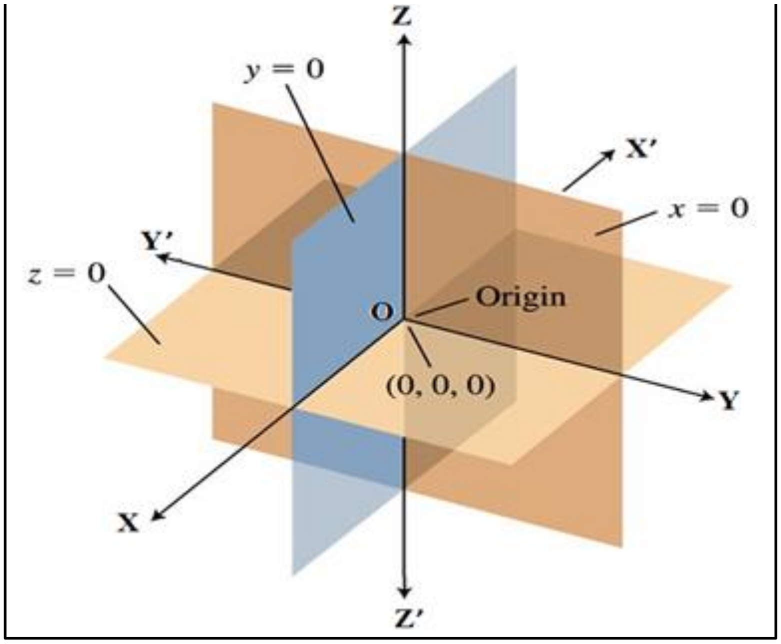
need one more dimension to tell the position of the airplane as it flies from one place to another at different times during its flight.

We live in three dimensional space, see electric bulb hanging from the ceiling of a room or the position of the central tip of the ceiling fan in a room, etc.

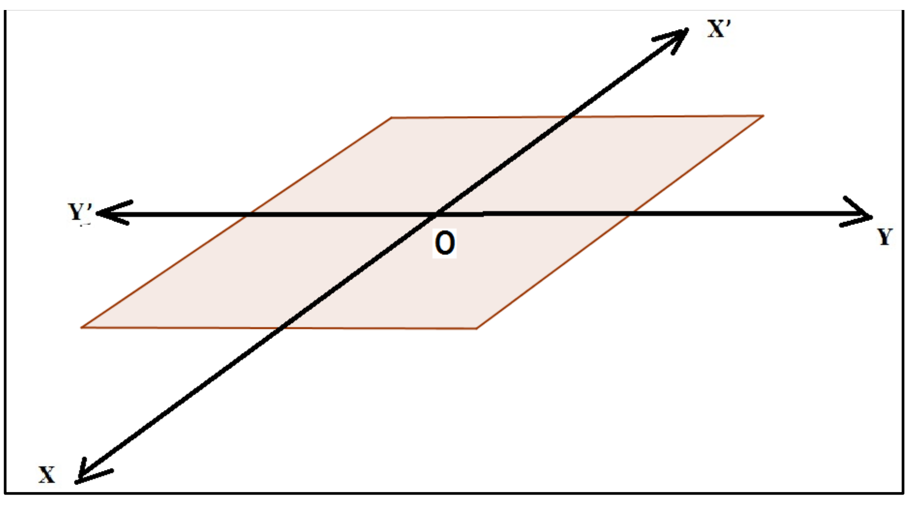
to locate the position of almost every object, we need not only the perpendicular distances of the object from two perpendicular walls of the room but also the height of the point from the floor of the room. Therefore, we need not only two but three numbers representing the perpendicular distances of the point from three mutually perpendicular planes, namely the floor and two adjacent walls of the room. The three numbers representing the three distances are called the coordinates of the point with reference to the three coordinate planes. Hence, a point in space has three coordinates. In this Module, we shall study the basic concepts of geometry in three dimensional space.

2. Coordinate Axes and Coordinate Planes in Three Dimensional Space:

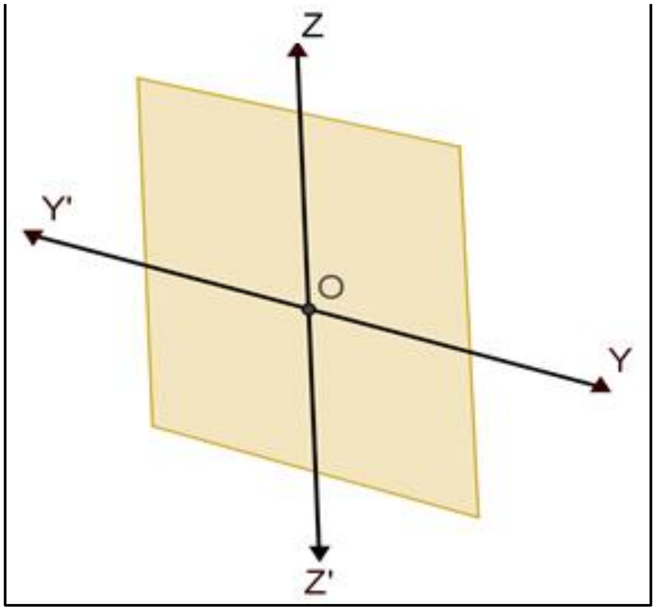
Let us consider three planes intersecting at a point O such that these three planes are mutually perpendicular to each other as shown below;



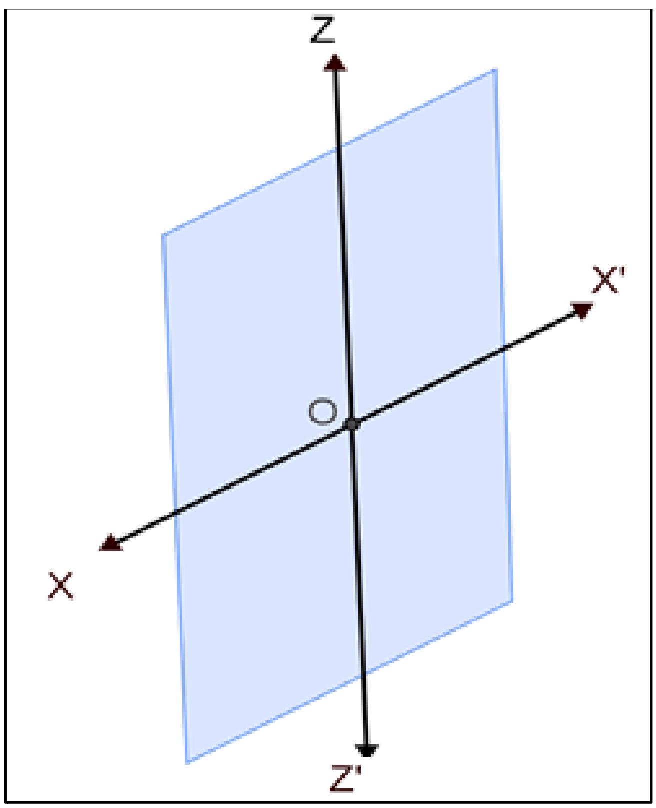
These three planes intersect along the lines X′OX, Y′OY and Z′OZ. These three lines are called the *x*-axis, y-axis and z-axis respectively. Observe that these lines are mutually perpendicular to each other. These lines constitute the rectangular coordinate system in three dimensional space.



The plane containing *x*-axis and y-axis is known as XY-plane, similarly plane containing y-axis and z-axis is known as YZ-plane

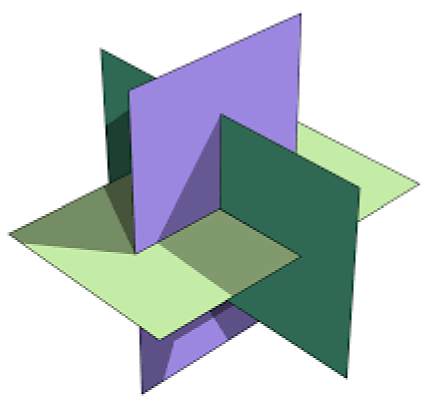


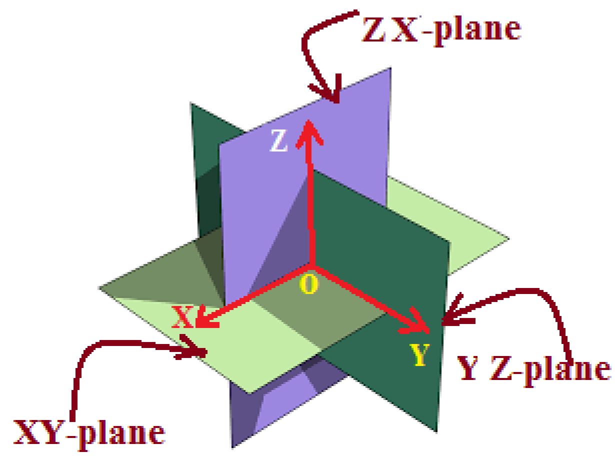
And the plane containing z-axis and *x*-axis is known as ZX-plane,



These three planes are known as the three coordinate planes. We take the XOY plane as the plane of the paper and the line Z′OZ as perpendicular to the plane XOY. If the plane of the paper is considered as horizontal, then the line Z′OZ will be vertical. The distances measured from XY-plane upwards in the direction of OZ are taken as positive and those measured downwards in the direction of OZ′ are taken as negative. Similarly, the distance measured to the right of ZX-plane along OY are taken as positive, to the left of ZX-plane and along OY′ as negative, in front of the YZ-plane along OX as positive and to the back of it along OX′ as negative. The point O is called the origin of the coordinate system.

The three coordinate planes divide the space into eight parts known as octants.

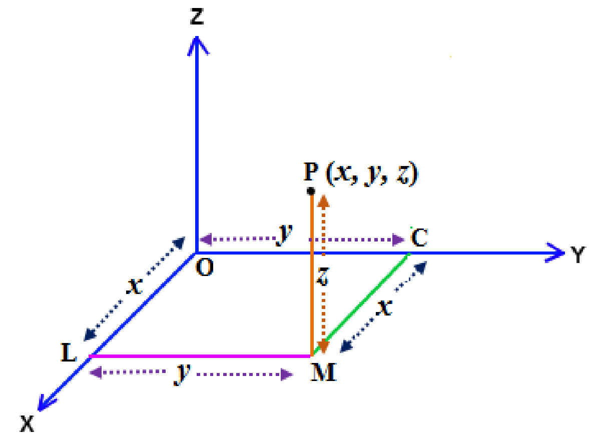




These octants could be named as XOYZ, X′OYZ, X′OY′Z, XOY′Z, XOYZ′, X′OYZ′, X′OY′Z′ and XOY′Z′ and denoted by I, II, III, ...and VIII octant respectively.

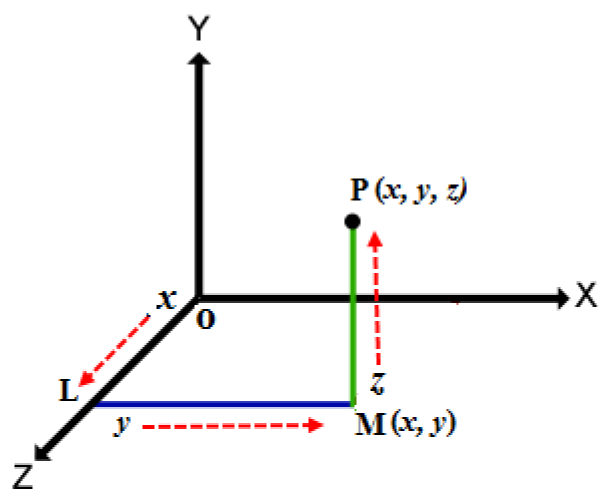
3. Coordinates of a Point in Space:

After choosing a fixed coordinate system in the space, consisting of three coordinate axes, three coordinate planes and the origin, let us now learn to associate three coordinates (x, y, z) to a given point in the space. Conversely, to a given triplet of three numbers (x, y, z), how to locate a point in the space.



Suppose, we have a point 'P' in space, we drop a perpendicular PM on the XY-plane with M as the foot of perpendicular, as shown above in the figure. From the point M, we draw a perpendicular ML to the x-axis, meeting x-axis at L. Let OL be x, LM be y and MP be z. Then x, y and z are x-coordinate, y-coordinate and z-coordinate of the point P respectively in the space. Observe that the point P (x, y, z) lies in the octant XOYZ i.e., I-octant, therefore, all x, y and z are positive. If P is in any other octant, then signs of x, y and z would change accordingly. Thus, to each point P in the space there corresponds an ordered triplet (x, y, z) of real numbers.

If any triplet (x, y, z) is given, then, we will first fix the point L on x-axis corresponding to x, then locate the point M in the XY-plane such that (x, y) are the coordinates of the point M in the XY-plane.



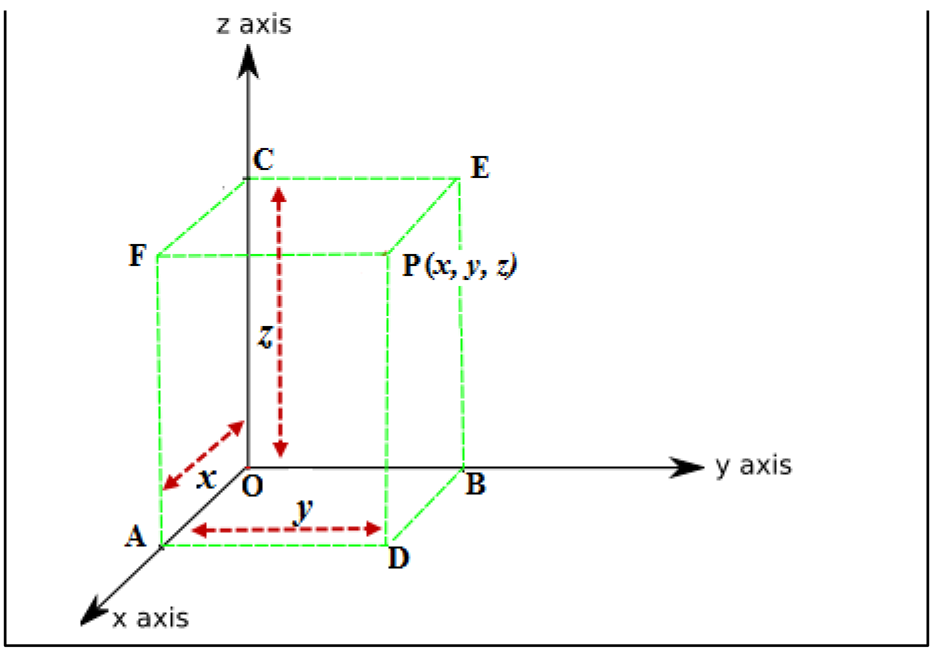
LM is perpendicular to the *x*-axis, i.e., parallel to the y-axis, hence, after reaching point M, we will draw a perpendicular MP to the XY-plane and locate on it the point P corresponding to the number '*z*'. The point P so obtained will have triplet (*x*, *y*, *z*) as its coordinates.

Thus we see that there is a one to one correspondence between the points in space and the ordered triplet (*x*, *y*, *z*) of real numbers. In other words we can say that to each point in space, we can associate a unique triplet (*x*, *y*, *z*) and to each triplet, we can associate a unique point in space.

Alternative Method:

We have learnt that, to associate coordinates (*x*, *y*, *z*) to a point in space, we have three mutually perpendicular coordinate planes, intersecting each other at three mutually perpendicular lines known as *x*-axis, y-axis and z-axis.

Hence, through point P in the space, we draw three planes parallel to the coordinate planes, meeting the *x*-axis, y-axis and z-axis at the points A, B and C, respectively. Let OA = *x*, OB = y and OC = z. Then, the point P will have the coordinates *x*, y and z and we write coordinates of point P as (*x*, y, z).



Conversely, if we are given the coordinates (*x*, y, z) of a point P, we locate three points A, B and C on the three coordinate axes as shown in the figure. Through the points A, B and C we draw planes parallel to the YZ-plane, ZX-plane and XY-plane respectively. The point of intersection of these three planes, ADPF, BDPE and CEPF is the point P, corresponding to the ordered triplet (*x*, y, z).

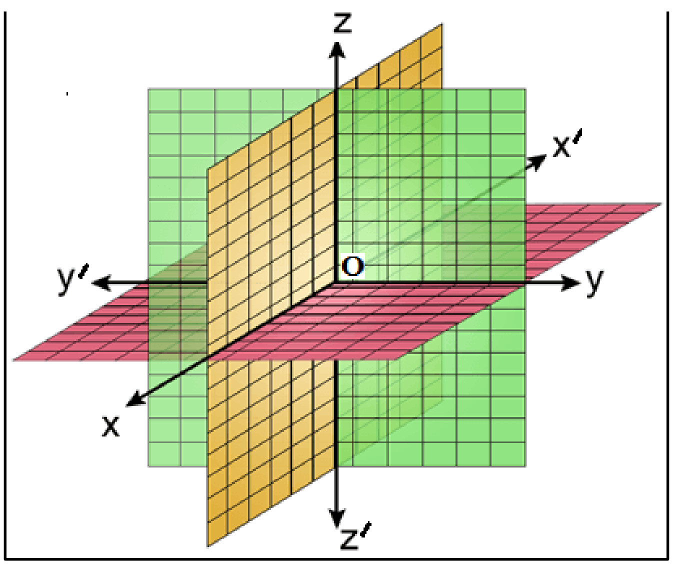
We observe that if P (*x*, y, z) is any point in the space, then *x*, y and z are perpendicular distances from YZ, ZX and XY planes, respectively.

Note:

The coordinates of the origin O are (0, 0, 0). The coordinates of any point on the *x*-axis will be (*x*, 0, 0), on the *y*-axis it will be (0, y, 0) and on the z-axis it will be as (0, 0, z). The coordinates of any point in the YZ-plane will be (0, y, z), any point in the ZX-plane will have the coordinates (*x*, 0, z) and the coordinates of any point in the XY-plane will be (0, y, z).

Remark:

The sign of the coordinates of any point determine the octant in which the point lies. The following picture of octants will help you to determine the signs of the coordinates in different quadrants.



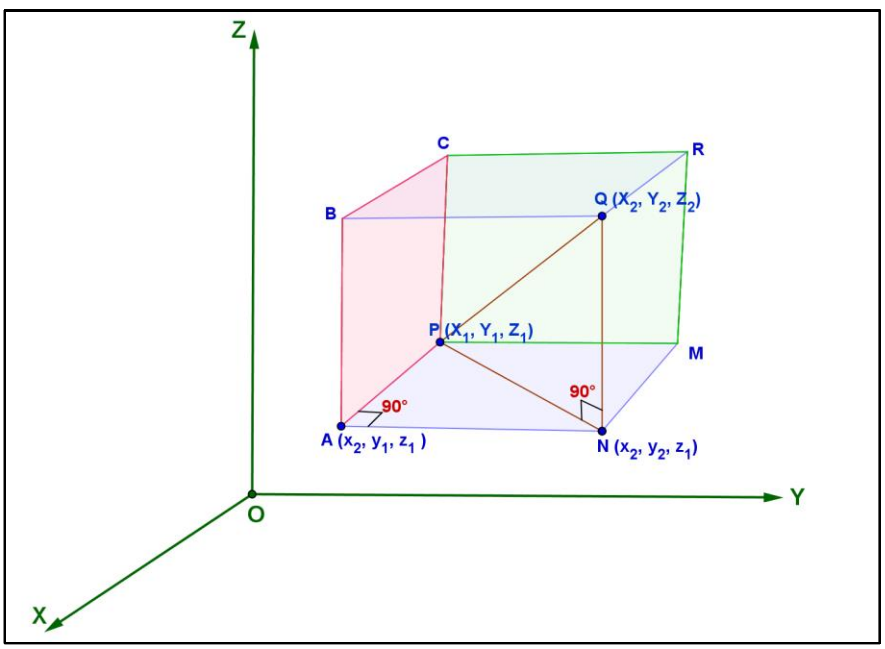
The following table shows us the signs of the coordinates in eight different octants.

Octants / Coordinates	I	II	III	IV	V	VI	VII	VIII
x	+	–	–	+	+	–	–	+
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

4. Distance between two points:

We have already learnt formula to find the distance between two points in two-dimensional coordinate system. Let us now extend this study to three-dimensional system.

Let P(x_1, y_1, z_1) and Q (x_2, y_2, z_2) be two points in three dimensional space, referred to a system of rectangular axes OX, OY and OZ. Through the points P and Q, planes parallel to the coordinate planes are drawn, to form a rectangular parallelopiped with PQ as one diagonal, shown in figure above.



Observe, AN is parallel to y-axis, hence, ∠PAN is a right angle, it follows that, in triangle PAN,

$$PN^2 = PA^2 + AN^2 \quad \ldots\ldots\ldots\ldots\ldots..(i)$$

Now, since ∠PNQ is a right angle, therefore, in triangle PNQ,

$$PQ^2 = PN^2 + NQ^2 \quad .... (ii)$$

From (i) and (ii), we have,

$$PQ^2 = PA^2 + AN^2 + NQ^2 \quad \ldots\ldots\ldots\ldots(iii)$$

Now, PA = $x_2 - x_1$, AN = $y_2 - y_1$ and NQ = $z_2 - z_1$

Hence, $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

Therefore,

$$PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

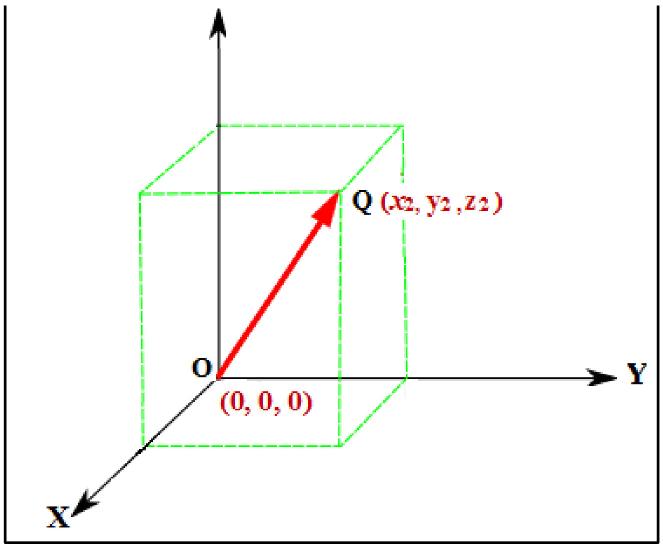
This is the required formula to find the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2).

Special case:

In particular, if $x_1 = y_1 = z_1 = 0$, i.e., point P is at origin O,

then, the distance of the point Q (x_2, y_2, z_2) from the origin is;

$$OQ = \sqrt{x_2^2+y_2^2+z_2^2}$$



5. Example:

In which octant the folowing points lie ;

(i) (–3, 1, –2), (ii) (–1, 2, 3), (iii) (2, 6, 8), (iv) (– 6, –1, –2).

Solution:

(i) (–3, 1, –2) lies in octant OX'YZ', because, x and z are negative and y is positive.

(ii) (–1, 2, 3) lies in the octant OX'YZ, since x is negative and y and z both are positive.

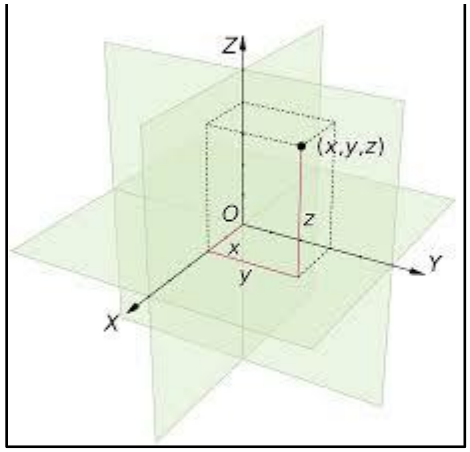
(iii) (2, 6, 8) lies in the octant OXYZ, because, all the co-ordinates are positive.

(iv) (– 6, –1, –2) lies in the octant OX'Y'Z', because, *x*, *y* and *z* are all Negative.

Example:

What is the distance of the point (3, 4, 5) from YZ-plane?

Solution:



From the figure, it is clear that, the distance of the point (x, y, z) from YZ plane is 'x', therefore, the distance of the point (3, 4, 5) from

YZ-plane is '3'.

Example:

Find the distance between the points P (–2, 1, –3) and Q (4, 3, – 6). **Solution:**

The distance PQ between the points P (–2, 1, –3) and Q (4, 3, – 6) is

$$PQ = \sqrt{\left(4-(-2)\right)^2+(3-1)^2+\left(-6-(-3)\right)^2}$$

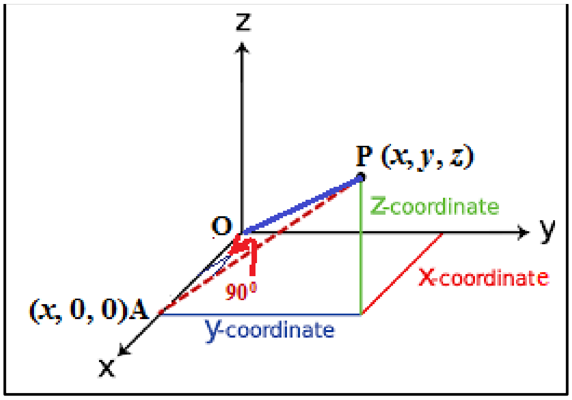
$$= \sqrt{36+4+9} = \sqrt{49} = 7$$

Example:

If A, B, C are the feet of perpendiculars from point P (–5, 3, 7) on the x, y, and z-axis respectively. Find the coordinates of the points A, B, C.

Solution:

From the following figure, it is obvious that, y and z coordinates of the point A, the foot of perpendicular from point P(x, y, z) on x-axis, will be zero,

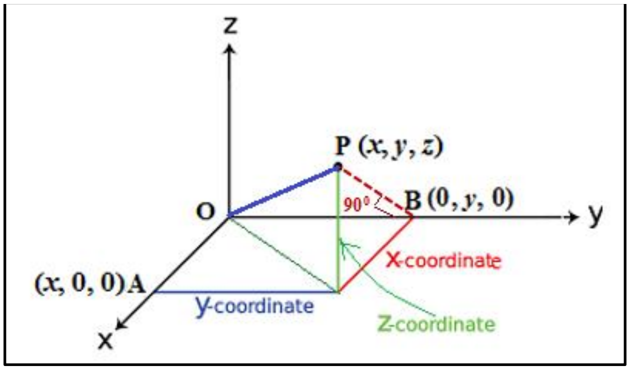


hence coordinates of point A, the foot of perpendicular from given point P (–5, 3, 7) on the x-axis, will be (–5, 0, 0),

the perpendicular distance of point P from x-axis,

$$\text{PA} = \sqrt{y^2 + z^2} = \sqrt{9 + 49} = \sqrt{58}$$

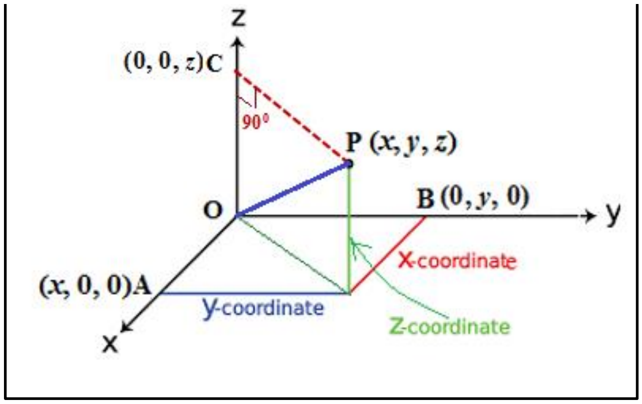
similarly coordinates of point B, the foot of perpendicular from given point P (–5, 3, 7) on the y-axis, will be (0, 3, 0),



the perpendicular distance of point P from y-axis,

$$\text{PB} = \sqrt{z^2 + x^2} = \sqrt{49 + 25} = \sqrt{74}$$

and the foot of perpendicular from given point P (–5, 3, 7) on the z-axis, will be (0, 0, 7),



the perpendicular distance of point P from z-axis,

$$PC = \sqrt{x^2 + y^2} = \sqrt{25 + 9} = \sqrt{34}$$

Example:

Prove that the points (–2, 4, –3), (4,–3, – 2) and (– 3,

– 2, 4) are the vertices of an equilateral triangle.

Solution:

Let the vertices of the triangle be A, B and C, using distance formula let us find the distances AB, BC and AC;

We know that, distance between two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) is given by the formula,

$$PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Hence,

$$AB = \sqrt{(4-2)^2 + (-3-4)^2 + (-2-3)^2}$$
$$= \sqrt{36 + 49 + 1} = \sqrt{86}$$

$$BC = \sqrt{(-3-4)^2 + (-2-3)^2 + (4-2)^2}$$
$$= \sqrt{49 + 1 + 36} = \sqrt{86}$$

$$AC = \sqrt{(-3+2)^2 + (-2-4)^2 + (4+3)^2}$$

$$= \sqrt{1+36+49} = \sqrt{86}$$

Thus, AB = BC = AC, hence the three sides of the triangle are equal,

the triangle is an equilateral triangle.

Example:

Find the equation of the set of points P such that $PA^2 + PB^2 = 2k^2$, where A and B are the points (3, 4, 5) and (–1, 3, –7), respectively.

Solution:

Let the coordinates of point P be (x, y, z).

Here, $PA^2 = (x-3)^2 + (y-4)^2 + (z-5)^2$

$PB^2 = (x+1)^2 + (y-3)^2 + (z+7)^2$

By the given condition $PA^2 + PB^2 = 2k^2$,

Hence we have,

$$(x-3)^2 + (y-4)^2 + (z-5)^2 + (x+1)^2 + (y-3)^2 + (z+7)^2 = 2k^2$$

Solving, we get,

$$2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z = 2k^2 - 109$$

above equation is the required equation.

6. Summary:

1) In three dimensions, the coordinate axes of a rectangular Cartesian coordinate system are three mutually perpendicular lines. These axes are called the x, y and z-axes.

2) The three planes determined by the pair of axes are the coordinate planes, called XY, YZ and ZX-planes.

3) The three coordinate planes divide the space into eight parts known as octants.

4) The coordinates of a point P in three dimensional geometry is always written in the form of triplet like (x, y, z). Here x, y and z are the distances from the YZ, ZX and XY-planes.

5) (i) Any point on x-axis is of the form $(x, 0, 0)$ (ii) Any point on y-axis is of the form $(0, y, 0)$ (iii) Any point on z-axis is of the form $(0, 0, z)$.

6) Distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$