1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 02 (Class XI, Semester - 2)
Module Name/Title	Conic Sections-Part 4
Module I	kemh_21104
Pre-requisites	Basic knowledge of Conic Sections
Objectives	After going through this lesson, the learners will be able to do the following: • Hyperbola • Eccentricity • Latus Rectum • Standard Equation of Hyperbola
Keywords	Hyperbola, Eccentricity, Latus Rectum, Standard equation of Hyberbola

2. Development Team

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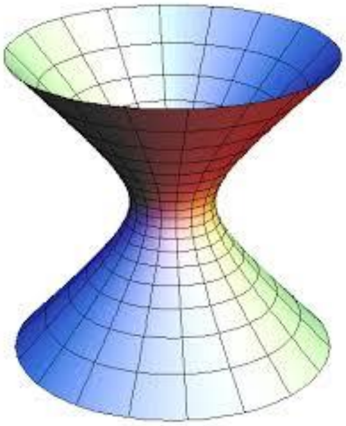
1. Introduction:

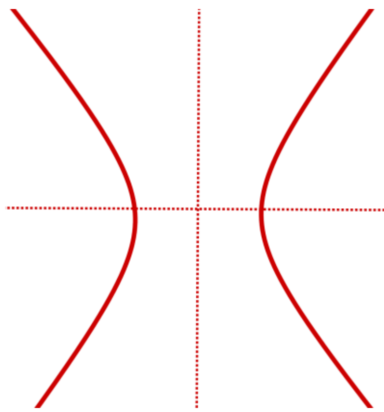
Observe the following figure, a very common example from our day to day life, the light rays coming out from the lamp-shade are symmetrical about the lamp shade.

Let us consider some more shapes from our real life situations,

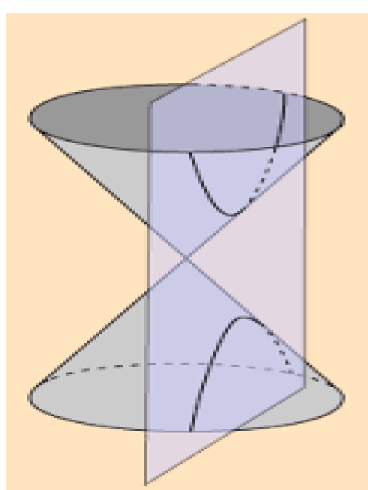


Observe these huge cooling towers,

If we try to figure out the shapes of these objects, we get the followig curve,

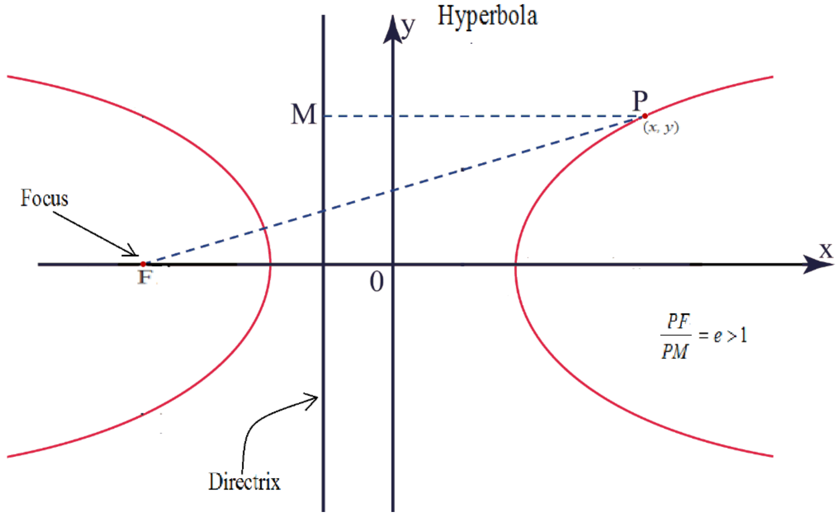


The curve is known as hyperbola, you have learnt earlier that it is a conic section and is obtained when a plane cuts both the nappes of a hollow double cone.

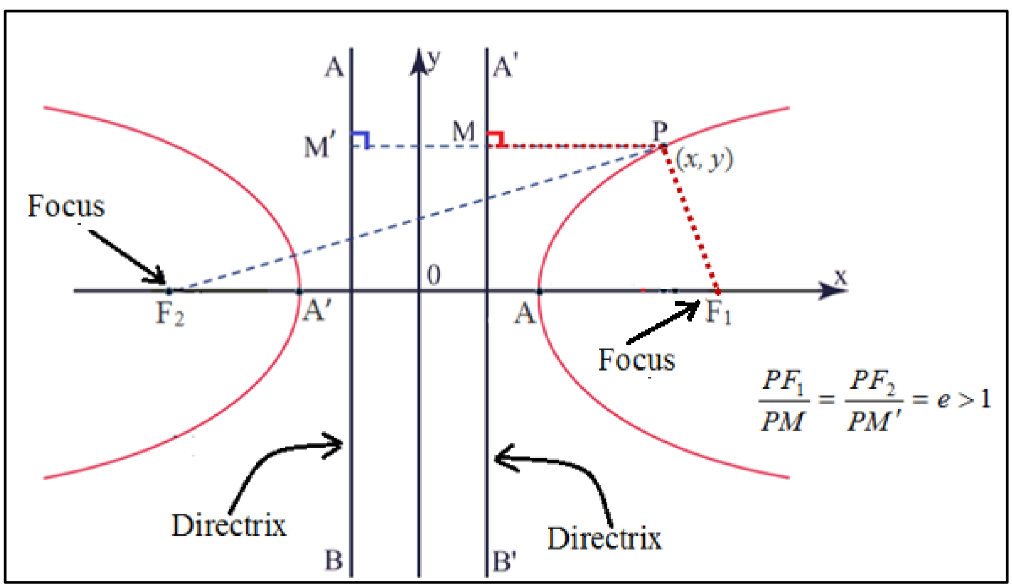


2. Hyperbola

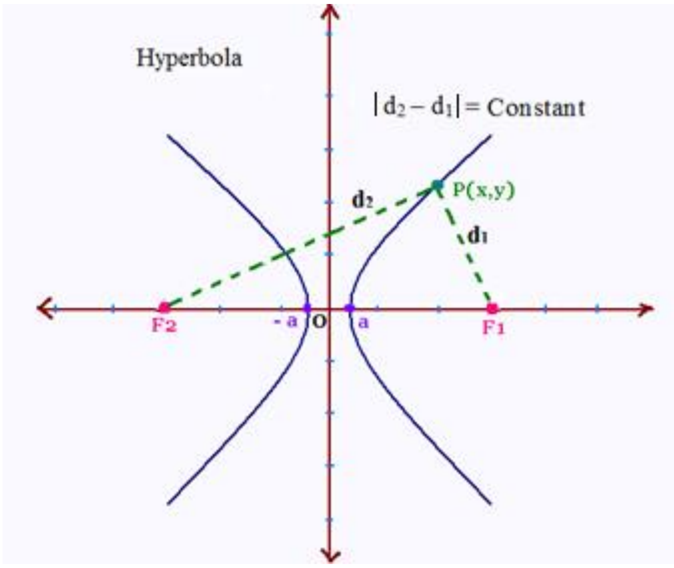
Just like ellipse, hyperbola is also locus of a point, which moves in a plane such that the ratio of its distance from a fixed point to its distance from a fixed straight line in the same plane is always constant, in case of hyperbola this ratio is always greater than one. The ratio is known as eccentricity of the hyperbola.



The fixed point is called focus and the fixed straight line is known as directrix. Hyperbola has two foci and two directrices.



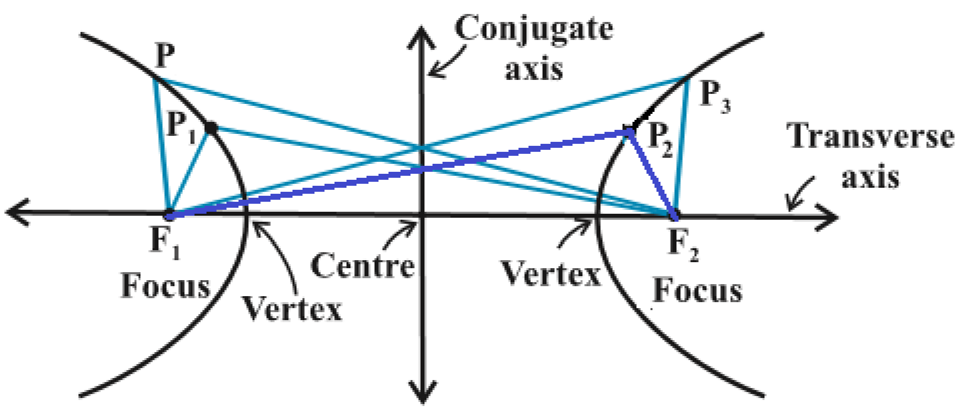
In a hyperbola, the difference of the distances of a point P, moving on the hyperbola, from two fixed points in the same plane is always constant.



These two points are the foci of the hyperbola.

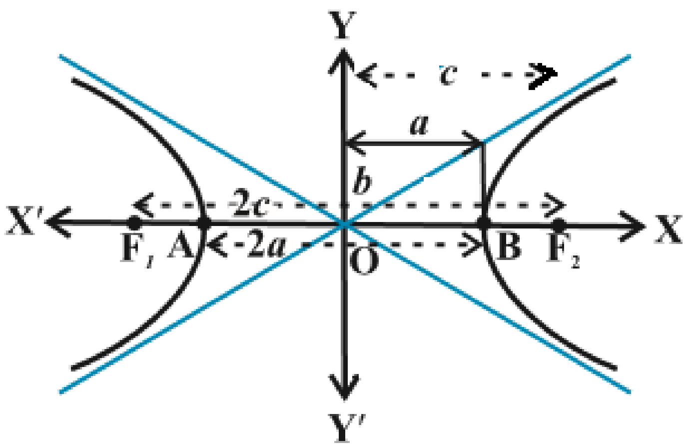
Definition:

A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.



$$P_1F_2 - P_1F_1 = P\ F_2 - P\ F_1 = P_3F_1 - P_3F_2 = P_2F_1 - P_2F_2 = \text{Constant}$$

The two fixed points are called the foci of the hyperbola. The mid-point of the line segment joining the foci is called the centre of the hyperbola. The line through the foci is called the transverse axis and the line through the centre and perpendicular to the transverse axis is called the conjugate axis. The points at which the hyperbola intersects the transverse axis are called the vertices of the hyperbola.



The distance between two foci is denoted by 2c, the distance between two vertices is known as the length of the transverse axis and is denoted by 2a, we define the quantity b as,

$$b = \sqrt{c^2 - a^2}$$

2b is called the length of the conjugate axis.

We know that the difference of a point P on the curve, from two fixed points in the plane is always a constant, let us find the value of this constant, $PF_2 - PF_1$.

Let us take the point at vertices A and B, then,

$BF_1 - BF_2 = AF_2 - AF_1$ (by the definition of the hyperbola)

$(BA + AF_1) - BF_2 = (AB + BF_2) - AF_1$

i.e., $2\,AF_1 = 2\,BF_2$

or $AF_1 = BF_2$ ……………(i)

When point P is at vertex B, we get

$PF_1 - PF_2 = BF_1 - BF_2$

$= (BA + AF_1) - BF_2$

$= BA = 2a$ ……………(ii) using (i)

Hence, if P is any point on the hyperbola, then the difference of its distances from foci is equal to 2a, which is equal to the length of the transverse axis.

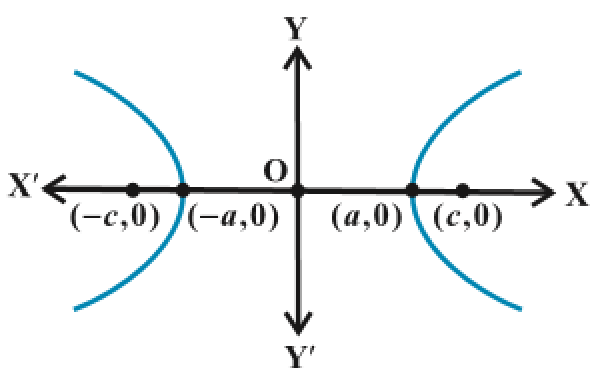
3. Eccentricity:

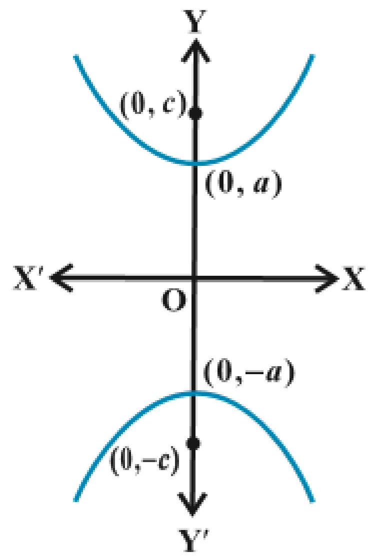
Just like an ellipse, the ratio, $e = \frac{c}{a}$ is the eccentricity of the hyperbola. Since $c \geq a$, the eccentricity is never less than one.

In terms of the eccentricity, the foci are at a distance of ae from the centre.

4. Standard equation of Hyperbola:

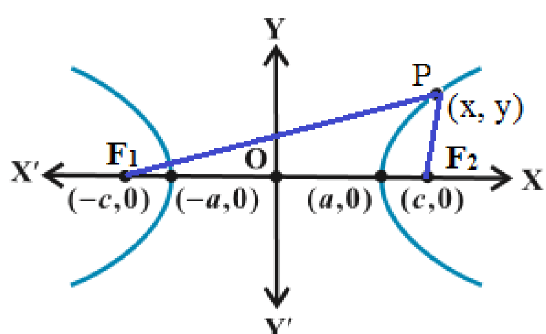
The equation of a hyperbola is simplest if the centre of the hyperbola is at the origin and the foci are either on the x-axis or y-axis. The two such possible orientations are shown below;





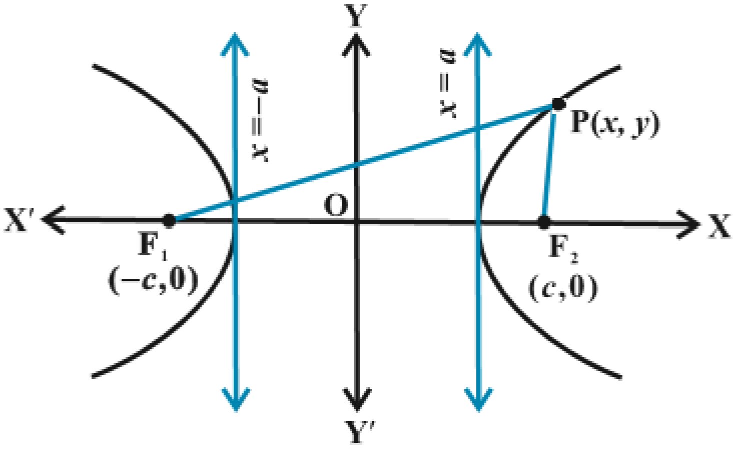
Let us derive the equation of the hyperbola with foci on *x*-axis and centre at origin.

Let F_1 and F_2 be the foci and O be the mid-point of the line segment F_1F_2. Let O be the origin and the line through O and F_2 be the positive x-axis and that through F_1 be the negative *x*-axis. The line through O, perpendicular to the *x*-axis be the y-axis. Let the coordinates of F_1 be $(-c, 0)$ and F_2 be $(c, 0)$.



Let P(x, y) be any point on the hyperbola and the difference of the distances from P to the foci be 2a.

i.e., $| PF_1 – PF_2 | = 2a$



Using the distance formula, we have

$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=2a$$

i.e., $$\sqrt{(x+c)^2+y^2}=2a+\sqrt{(x-c)^2+y^2}$$

Squaring both side, we get

$$(x+c)^2+y^2=4a^2+4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

Simplifying, we get

$$\frac{cx}{a}-a=\sqrt{(x-c)^2+y^2}$$

……..(1)

Conversely, let us take a point P(x, y) which satisfies the above equation with $0 < a < c$, then from equation (1) we get,

$$y^2 = b^2 \left(\frac{x^2 - a^2}{a^2} \right)$$

But by distance formula,

$$PF_1 = \sqrt{(x+c)^2 + y^2}$$

Substituting for y^2, we get,

$$PF_1 = \sqrt{(x+c)^2 + b^2 \left(\frac{x^2 - a^2}{a^2} \right)}$$

Simplifying and substituting, $b^2 = c^2 - a^2$, we get,

$$PF_1 = \sqrt{\left(a + \frac{c}{a} x \right)^2}$$

$$= a + \frac{c}{a} x$$

Similarly we get, $PF_2 = a - \frac{c}{a} x$

In hyperbola c > a; since P is to the right of the line x = a, so x > a,

Therefore, $\frac{c}{a} x > a$; $a - \frac{c}{a} x$ becomes negative,

Thus, $PF_2 = \frac{c}{a} x - a$

Using these values of PF_1 and PF_2 we get,

$$PF_1 - PF_2 = a + \frac{c}{a}x - \left(\frac{cx}{a} - a\right)$$

$$= a + \frac{c}{a}x - \frac{cx}{a} + a$$

$$= 2a$$

If P is to the left of the line $x = -a$, then we get,

$$PF_1 = -\left(a + \frac{c}{a}x\right) \qquad \text{and} \qquad PF_2 = a - \frac{c}{a}x$$

In this case we get, $PF_2 - PF_1 = 2a$,

Hence, any point that satisfies the equation,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

lies on the hyperbola.

Thus the equation of the hyperbola with centre at origin (0,0) and transverse axis along x-axis is,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

Note: A hyperbola in which a = b is called rectangular hyperbola.

Discussion: From the equation of the hyperbola we notice that, for every point (x, y) on the hyperbola,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

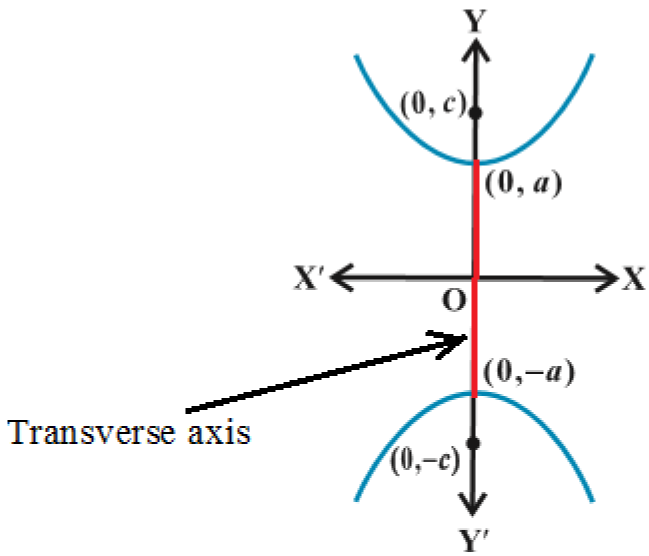
we get,

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2}$$, which is always, ≥ 1,

i.e., $\left|\frac{x}{a}\right| \geq 1$, we have $x \leq -a$ or $x \geq a$

Therefore, no portion of the curve lies between the lines $x = +a$ and $x = -a$, (i.e. no real intercept on the conjugate axis).

Similarly, we can derive the equation of the hyperbola with centre at origin and transverse axis along y-axis,



In this case the equation of the hyperbola is,

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad \text{……………(3)}$$

Equations (2) and (3) are known as the standard equations of hyperbolas.

Note: The standard equations of hyperbolas have transverse and conjugate axes as the coordinate axes and the centre at the origin.

From the standard equations of hyperbolas, we have the following observations:

1. Hyperbola is symmetric with respect to both the axes, since if (x, y) is a point on the hyperbola, then (– x, y), (x, – y) and (– x, – y) are also points on the hyperbola.

2. The foci are always on the transverse axis. The positive term gives the transverse axis.

Example:

1) Equation,

$$\frac{x^2}{9}-\frac{y^2}{16}=1$$ is similar to equation $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Here, term containing x^2 is positive, therefore, transverse axis is along x-axis and $a^2 = 9$,

Therefore, $a = 3$ and length of transverse axis $= 2a = 6$.

2) Equation,

$$\frac{y^2}{25}-\frac{x^2}{16}=1$$ is similar to equation $$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

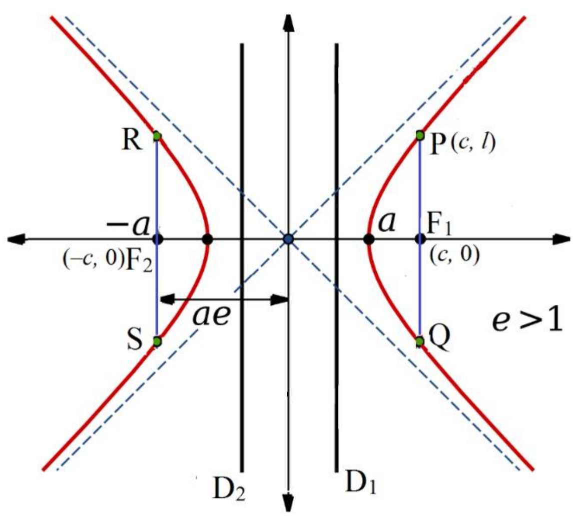
In this case, term containing y^2 is positive, therefore, transverse axis is along y-axis and $a^2 = 25$,

Therefore, $a = 5$ and length of transverse axis $= 2a = 10$.

5. Latus rectum:

Latus rectum of hyperbola is a line segment perpendicular to the transverse axis through any of the foci and whose end points lie on the hyperbola.

In the following figure,



PQ and RS are the line segments, perpendicular to the transverse axis, having their end points on hyperbola and are passing through the foci of the hyperbola, hence, they are the latus rectum of the hyperbola.

Let length, $PF_1 = l$, then, coordinates of point P are (c, l). Since point P lies on the hyperbola, hence,

$$\frac{c^2}{a^2} - \frac{l^2}{b^2} = 1$$

Therefore, $$\frac{l^2}{b^2} = \frac{c^2}{a^2} - 1$$

or $$l^2 = b^2\left(\frac{c^2 - a^2}{a^2}\right) = b^2\left(\frac{b^2}{a^2}\right) \quad (\text{Since, } b^2 = c^2 - a^2)$$

hence, $$l = \frac{b^2}{a}$$

Length of line segment RS = Length of line segment PQ

$$= 2l = \frac{2b^2}{a}$$

= Length of latus rectum

6. Example:

Find the coordinates of the foci, vertices, eccentricity and the length of the latus rectum of the following hyperbola,

$$\frac{y^2}{9} - \frac{x^2}{27} = 1$$

Solution:

Comparing the given equation with standard equation,

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

We get, $a = 3$, $b = 3\sqrt{3}$

and $$c = \sqrt{3^2 + \left(3\sqrt{3}\right)^2} = \sqrt{9 + 64} = \sqrt{36}$$

$$= 6$$

Therefore, the coordinates of the foci are $(0, \pm 6)$

Since $a = 3$, the coordinates of vertices are $(0, \pm 3)$.

The eccentricity, $e = \frac{c}{a} = \frac{6}{3} = 2$

The length of the latus rectum = $\frac{2b^2}{a}$= $\frac{2\times27}{3}$= 18

Example:

Find the coordinates of the foci, vertices, eccentricity and the length of the latus rectum of the following hyperbola,

$$16x^2 - 9y^2 = 576$$

Solution:

Let us first write the equation in the standard form, dividing the equation by 576 on both sides, we have

$$\frac{16x^2}{576} - \frac{9y^2}{576} = 1 \quad \text{or} \quad \frac{x^2}{36} - \frac{y^2}{64} = 1$$

We can write it as, $\frac{x^2}{6^2} - \frac{y^2}{8^2} = 1$,

comparing it with the standard equation,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 ,$$

We get, a = 6, b = 8

and $c = \sqrt{a^2 + b^2}$= $\sqrt{6^2 + 8^2}$

= $\sqrt{36 + 64}$ = $\sqrt{100}$ = 10.

Therefore, the coordinates of the foci are (± 10, 0) and since a = 6, the coordinates of vertices are (± 6, 0).

Now, the eccentricity, e = $\frac{c}{a}$ = $\frac{10}{6}\frac{5}{3}$

The length of the latus rectum = $\frac{2b^2}{a}$= $\frac{2\times8^2}{6}$= $\frac{64}{3}$

Example:

Find the equation of a hyperbola whose vertices are at (0, –3) and (0, 3) and one of the foci at (0 , 5).

Solution:

Since vertices of the parabola are at (0, –3) and (0, 3) and one of the foci at (0, 5), hence, centre of the hyperbola is at (0, 0), a = 3 and c = 5.

Transverse axis is along y-axis, therefore, equation of the hyperbola is;

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

We know that, $b^2 = c^2 - a^2$, therefore, $b^2 = 5^2 - 3^2$

Or $b^2 = 25 - 9 = 16 = 4^2$,

The required equation of the ellipse is,

$$\frac{y^2}{3^2} - \frac{x^2}{4^2} = 1 \quad \text{or} \quad \frac{y^2}{9} - \frac{x^2}{16} = 1$$

Example:

Find the equation of a hyperbola which has transverse axis along x-axis and is of length 12. The eccentricity of the hyperbola is $\frac{4}{3}$ and centre at origin.

Solution:

Since centre of the hyperbola is at origin and transverse axis along x-axis, therefore, the equation of the hyperbola will be of the type,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{..................(i)}$$

Length of the transverse axis = $2a = 12$

Therefore, $a = 6$ and $e = \frac{4}{3}$, $c = ae = 6 \text{ x} \frac{4}{3} = 8$

$$b^2 = c^2 - a^2 = 8^2 - 6^2 = 64 - 36 = 28$$

Substituting the values of a^2 and b^2, the required equation is of the hyperbola is,

$$\frac{x^2}{36} - \frac{y^2}{28} = 1$$

Example:

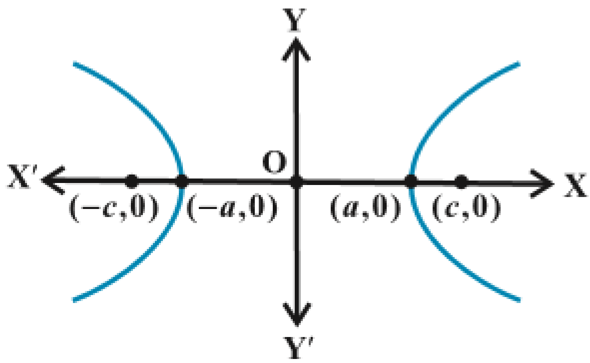
Find the equation of the hyperbola with centre at (0, 0), foci on x-axis, focal length equal to 34 and the distance from one focus to the closest vertex is 2.

Solution:

Given that focal length equal is to 34,

Therefore, $2c = 34$ or $c = 17$,

The distance from one focus to the closest vertex is 2.



Therefore, $a = c - 2$

$= 17 - 2 = 15$,

Now, $b^2 = c^2 - a^2 = (17)^2 - (15)^2 = 64$,

Hence required equation of the ellipse is;

$$\frac{x^2}{15^2} - \frac{y^2}{8^2} = 1$$

or $$\frac{x^2}{225} - \frac{y^2}{64} = 1$$

Example:

Find the equation of a hyperbola whose foci are at (0, – 4) and (0, 4) and the latus rectum is of length 12.

Solution:

Since foci are (0, ± 4), therefore, $c = 4$

Length of the latus rectum = $\frac{2b^2}{a} = 12$,

Hence, $b^2 = 6a$

We know, $c^2 = a^2 + b^2$,

Therefore, $a^2 = c^2 - b^2$

Or $a^2 = 16 - 6a$

$a^2 + 6a - 16 = 0$

Solving we get, $a = -8, 2$, but a cannot be negative, hence, $a = 2$,

Thus, $b^2 = 6a = 12$, and $a^2 = 4$,

Therefore, the equation of the required hyperbola is,

$$\frac{y^2}{4} - \frac{x^2}{12} = 1$$

Example:

Find equation of a hyperbola whose foci are at $(0, \pm\sqrt{10})$ and which passes through the point (2, 3)

Solution:

Since foci are at $(0, \pm\sqrt{10})$, therefore, transverse axis is along y-axis,

and $c = \sqrt{10}$, we know that, $c^2 = a^2 + b^2$, therefore, $10 = a^2 + b^2$

or, $a^2 = 10 - b^2$ ………………..(i)

The equation of the hyperbola is,

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

It passes through the point (2, 3), hence,

$$\frac{3^2}{a^2} - \frac{2^2}{b^2} = 1 \quad \text{or} \quad \frac{9}{a^2} - \frac{4}{b^2} = 1 \qquad \ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Solving (i) and (ii), we get, $b^2 = 5, -8$, but b^2 can not be negative,

Hence, $b^2 = 5$, and from (i) $a^2 = 5$,

Thus the required equation of the hyperbola is, $\frac{y^2}{5} - \frac{x^2}{5} = 1$

7. Summary

1. A hyperbola is a conic section which is obtained when a plane cuts both the nappes of a hollow double cone.
2. It is locus of a point, which moves in a plane such that the ratio of its distance from a fixed point to its distance from a fixed straight line in same plane is always greater than one. This ratio is known as eccentricity of the hyperbola.

3. The fixed point is called the focus and the fixed straight line is known as directrix of the hyperbola.

4. In a hyperbola, the difference of the distances of a point P, moving on the hyperbola, from two fixed points in the same plane is always constant. These two fixed points are the foci of the hyperbola.

5. A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.

6. The equation of a hyperbola with foci on the *x*-axis is,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

7. The equation of a hyperbola with foci on the y-axis is,

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

8. Latus rectum of hyperbola is a line segment perpendicular to the transverse axis through any of the foci and whose end points lie on the hyperbola.
9. Length of the latus rectum of the hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is $\frac{2b^2}{a}$
10. The eccentricity of a hyperbola is the ratio of the distances from the centre of the hyperbola to one of the foci and to one of the vertices of the hyperbola.
11. Eccentricity of different conic sections,
    i. For parabola, e = 1
    ii. For ellipse, e < 1
    iii. For hyperbola, e > 1
    iv. For circle, e = 0