1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 02 (Class XI, Semester - 2)
Module Name/Title	Conic Sections-Part 3
Module I	kemh_21103
Pre-requisites	Basic knowledge of Conic Sections
Objectives	After going through this lesson, the learners will be able to do the following: • Hyperbola • Eccentricity • Latus Rectum • Standard Equation of Hyperbola
Keywords	Hyperbola, Eccentricity, Latus Rectum, Standard equation of Hyberbola

2. Development Team

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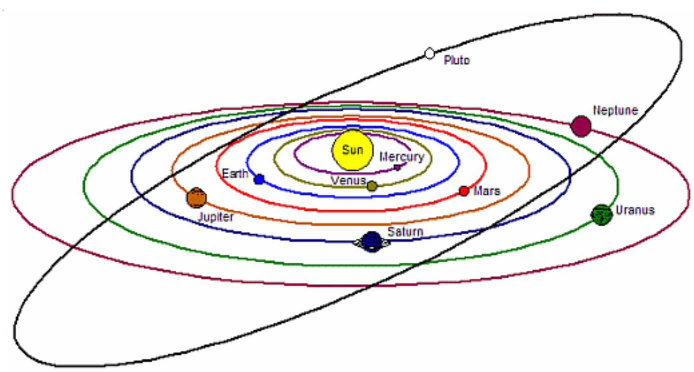
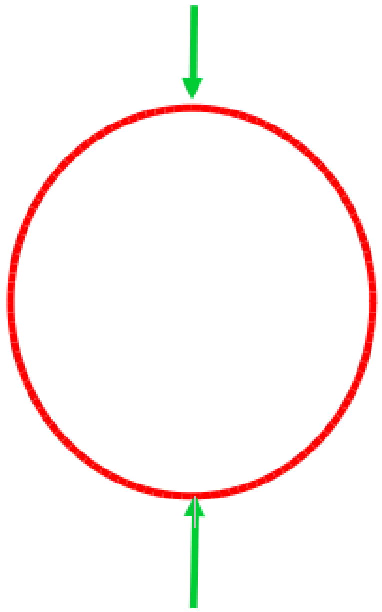
Table of Contents:

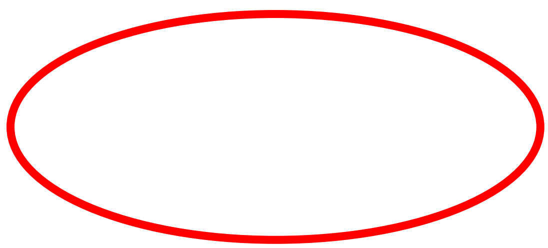
1. Introduction: From earlier classes you are quite familiar with solar system and know that planets move around sun in elliptical orbits.



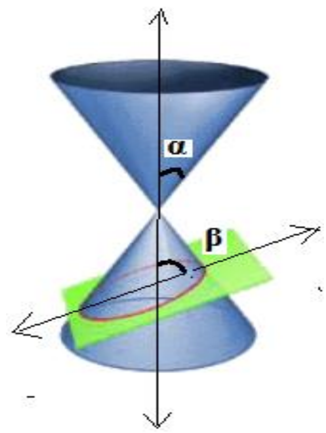
If we take a circle made up of thick wire and stretch it out as shown below, or press it inside from top and bottom as shown,



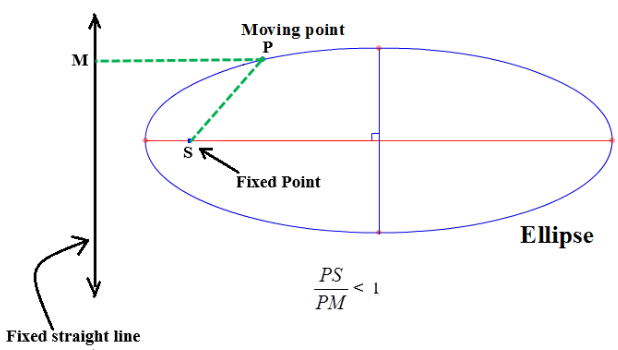
The circle deforms and takes the shape of an ellipse,



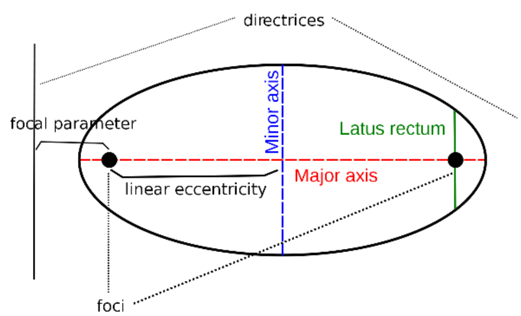
You have learnt in previous module on conic section that if a plane cuts a double cone at an angle β such that, $\alpha < \beta < 90^0$, the section is an ellipse, where, α is the semi vertical angle of the cone.



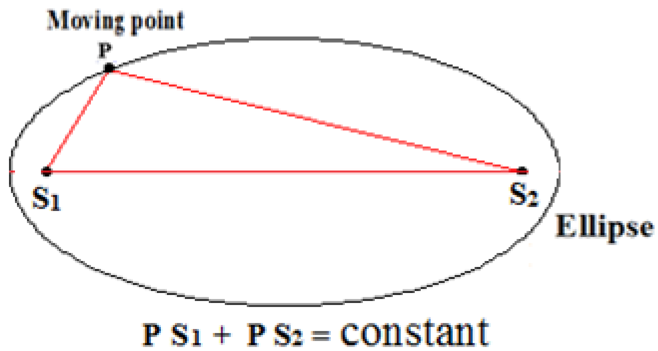
Again, if we find locus of a point which moves in a plane such that its distance from a fixed point in the plane bears a constant ratio with its distance from a fixed straight line in the same plane, then the curve so obtained is ellipse, if the ratio is less than one.



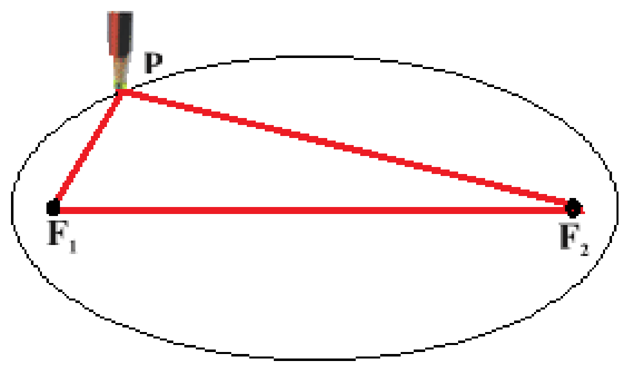
The fixed point is called focus and the fixed straight line is known as directrix. Ellipse has two foci and two **directrices**.



We have a remarkable property in ellipse, the sum of the distances of a point P moving on an ellipse, from to fixed points in the same plane is always constant.

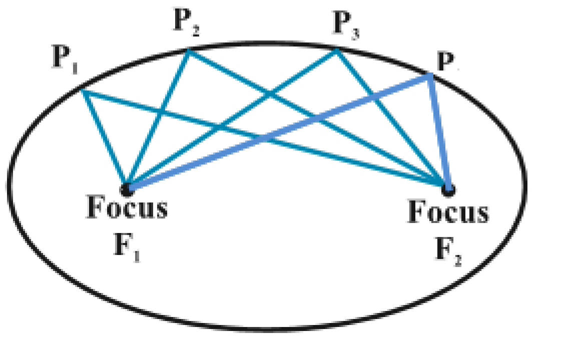


This property can be demonstrated through a simple experiment, Fasten a piece of string between two pins pressed into a board (the string to be taken longer than the distance between the pins). Move the pencil tip around, keeping the string tight, the path that is traced out by the pencil will be an ellipse.



Definition:

An ellipse is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant.



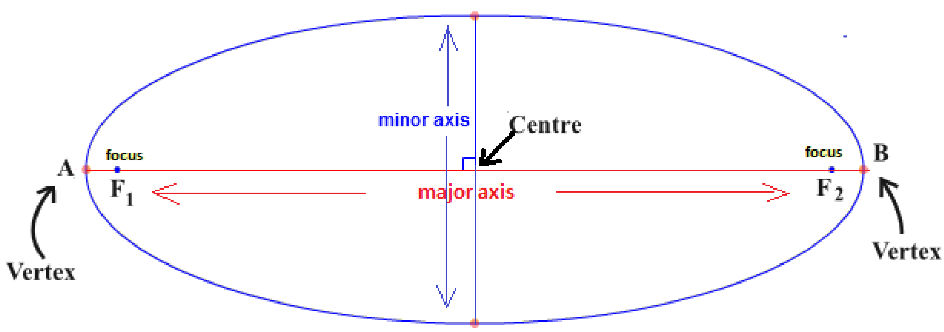
$$P_1F_1 + P_1F_2 = P_2F_1 + P_2F_2 = P_3F_1 + P_3F_2$$
$$= \cdots\cdots = P\,F_1 + P\,F_2$$

The two fixed points are called the foci of the ellipse. Foci is the plural of 'focus'.

Note:

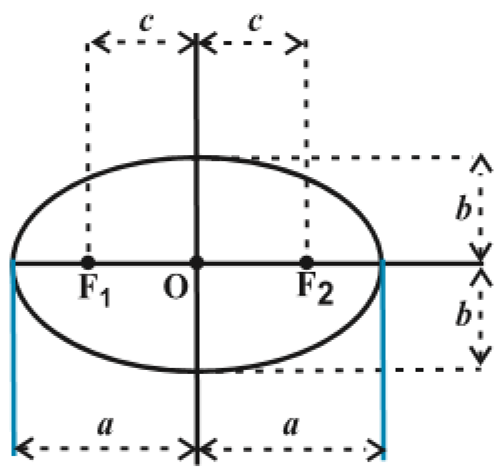
The constant which is the sum of the distances of a point on the ellipse from the two fixed points is always greater than the distance between the two fixed points.

The mid-point of the line segment joining the foci is called the centre of the ellipse.

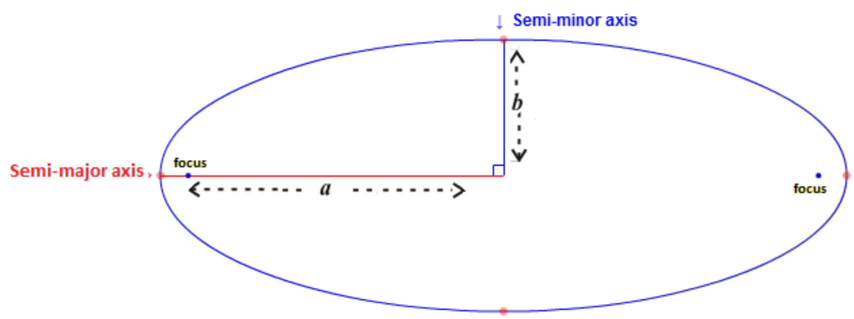


The line segment through the foci of the ellipse is called the major axis and the line segment through the centre and perpendicular to the major axis is called the minor axis. The end points of the major axis are called the vertices of the ellipse.

We denote the length of the major axis by 2a, the length of the minor axis by 2b and the distance between the foci by 2c, as shown below.

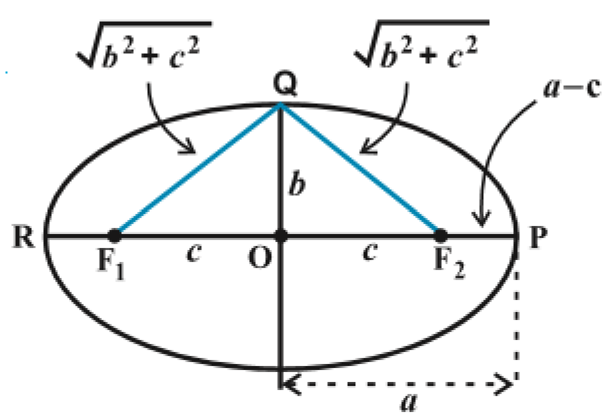


Thus, the length of the semi major axis is ‘a’ and semi-minor axis is ‘b’.



Relationship between semi-major axis, semi-minor axis and the distance of the focus from the centre of the ellipse:

Take a point P at one end of the major axis.



Sum of the distances of the point P to the foci is

$F_1 P + F_2P = (F_1O + OP) + (OP – OF_2)$

$= (c + a) + (a – c) = 2a$ …………..(i)

Take a point Q at one end of the minor axis. Sum of the distances from the point Q to the foci = $F_1Q + F_2Q$

$$= \sqrt{b^2 + c^2} + \sqrt{b^2 + c^2}$$

$$= 2\sqrt{b^2 + c^2} \qquad \text{…………..(ii)}$$

Since both P and Q lie on the ellipse. Therefore, according to definition of ellipse, both the distances, (i) and (ii) should be equal, hence, we have

$$2\sqrt{b^2 + c^2} = 2a$$

Or $$a = \sqrt{b^2 + c^2}$$

Squaring, $a^2 = b^2 + c^2$

Or $c = \sqrt{a^2 - b^2}$

2. Special cases of an ellipse:

In the equation,

$$c^2 = a^2 – b^2$$

if we keep '*a*' fixed and vary '*c*' from 0 to *a*,

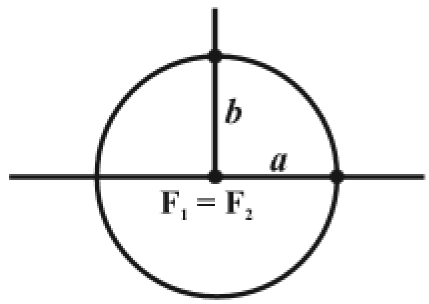
then '*b*' will vary from *a* to 0 and shape of the resulting ellipses will vary.

Case I:

When c = 0, distance between the foci will be zero i.e., both the foci will merge together with the centre of the ellipse and we will get,

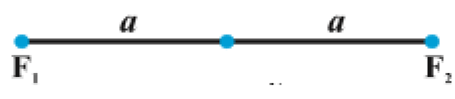
$a^2 = b^2$, i.e., $a = b$,

and thus the ellipse becomes a circle. So, we can say that a circle is a special case of an ellipse, when length of major axis is equal to the length of minor axis.



Case II:

When c = a, we get b = 0, i.e., length of semi-minor axis is zero. The ellipse reduces to the line segment F_1F_2 joining the two foci.



3. Eccentricity:

The ratio $\frac{c}{a}$ is called the eccentricity of the ellipse, it is denoted e,

Thus, $e = \frac{c}{a}$

Hence, the eccentricity of an ellipse is the ratio of the distances from the centre of the ellipse to one of the foci and to one of the vertices of

the ellipse.

Focus is at a distance of c from the centre, therefore, in terms of the eccentricity the focus is at a distance of ‘ae’ from the centre.

Again, in the introduction, we have seen that ellipse is the locus of a point which moves in a plane such that its distance from a fixed point in the plane (focus) bears a constant ratio to its distance from a fixed straight line in the same plane (directrix), if the ratio is less than one.

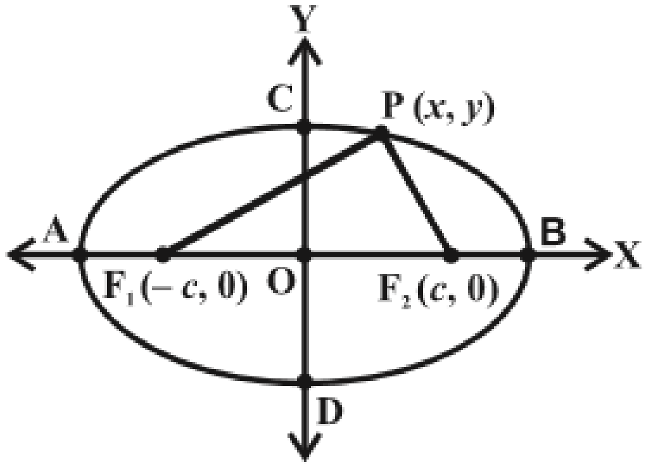
This ratio is eccentricity of the ellipse which is always less than unity.

$$e = \frac{c}{a} < 1 \quad \Rightarrow \quad c < a$$

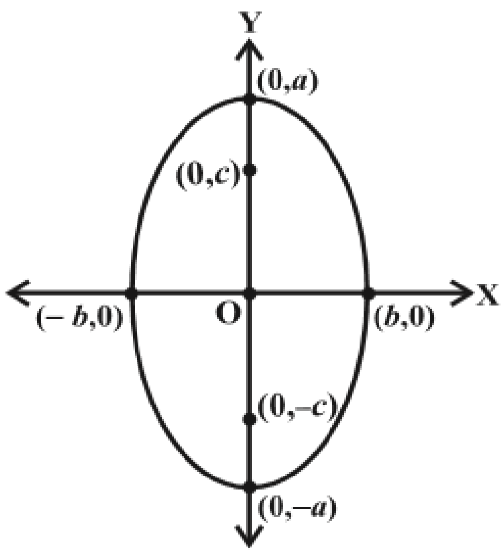
4. Standard equations of an ellipse:

The equation of an ellipse is simplest if the centre of the ellipse is at the origin and the foci are on one of the coordinate axes. The two such possible ellipses are shown below;

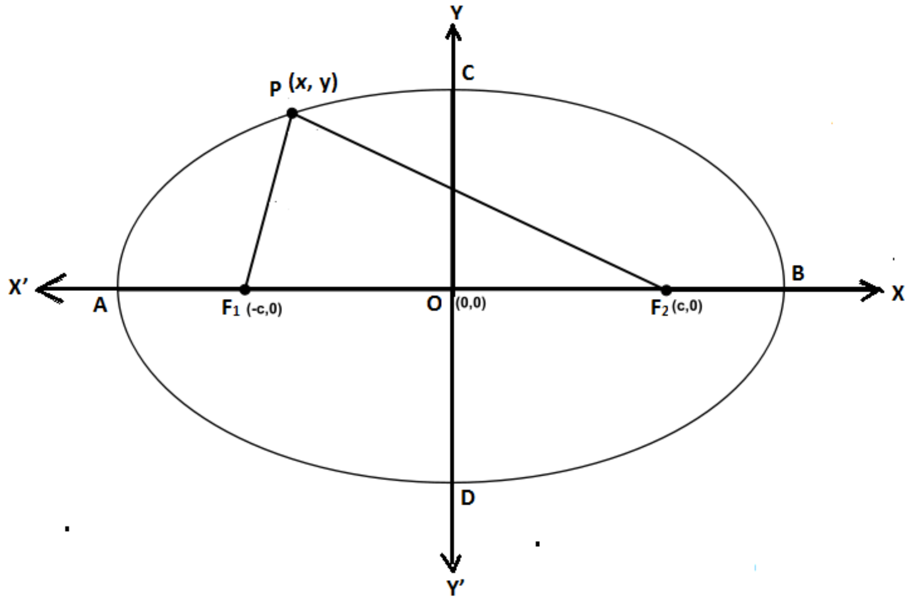
(a) Foci on x-axis



(b) Foci on y-axis



Let us derive the equation for the ellipse with foci on the x-axis and centre of the ellipse at origin. Let F_1 and F_2 be the foci and O be the mid-point of the line segment F_1F_2.



Let us take O as origin and the line from O through F_2 be the positive *x*-axis and that through F_1 as the negative *x*-axis. Let, the line through O perpendicular to the *x*-axis be the y-axis. Coordinates of F_1 be (– c, 0) and that of F_2 be (c, 0).

Let P(x, y) be any point on the ellipse and the sum of the distances of point P from the two foci be 2a, then, as per the definition of ellipse, we have,

$$PF_1 + PF_2 = 2a \qquad \ldots (1)$$

Using the distance formula, we have,

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

Squaring both the sides we get,

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

on simplification we get,

$$\sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x$$

Again squaring both the sides, we get,

$$(x + c)^2 + y^2 = 4a - 4a\sqrt{(x-c)^2 + (y)^2} + (x - c)^2 + y^2$$

or $$x^2 - 2cx + c^2 + y^2 = a^2 - 2cx + (\tfrac{c}{a}x)^2$$

or $$x^2 + c^2 + y^2 = a^2 + \frac{c^2}{a^2}x^2$$

or $$(a^2 - c^2)\,x^2 + a^2c^2 + a^2y^2 = a^4$$

or $$(a^2 - c^2)\,x^2 + a^2y^2 = a^4 - a^2c^2$$

or $$(a^2 - c^2)\,x^2 + a^2y^2 = a^2(a^2 - c^2)$$

dividing both the sides by $a^2(a^2 - c^2)$, we get,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

or $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (c^2 = a^2 - b^2)$$

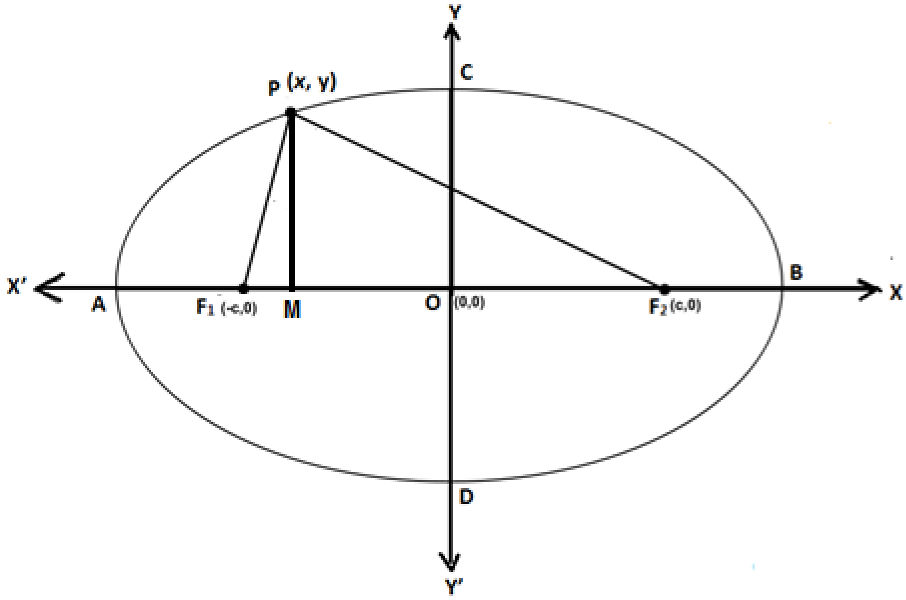
Hence any point on the ellipse satisfies the equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \ldots\ldots\ldots\ldots(2)$$

Conversely, let any point P (x, y) satisfy the equation (2), and $0 < c < a$, then,

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) \qquad \ldots\ldots\ldots\ldots(3)$$

From the following figure we have,



Therefore, $PF_2 = \sqrt{(x+c)^2 + y^2}$

$= \sqrt{(x+c)^2 + b^2\left(\frac{a^2 - x^2}{a^2}\right)}$ From equation (3)

$= \sqrt{(x+c)^2 + (a^2 - c^2)\left(\frac{a^2 - x^2}{a^2}\right)}$ $(\because b^2 = a^2 - c^2)$

Further simplifying we get,

$$\sqrt{(x+c)^2 + \left(1 - \frac{c^2}{a^2}\right)(x^2 - c^2)}$$

$$\sqrt{(x^2 + 2\text{cx} + c^2) + \left(a^2 - x^2 - c^2 + \frac{c^2x^2}{a^2}\right)}$$

$$= \sqrt{\left(a + \frac{cx}{a}\right)^2} = a + \frac{c}{a}x$$

…………. (4)

Similarly,

$$PF_1 = a - \frac{c}{a}x \qquad \text{............ (5)}$$

Adding (4) and (5), we get,

$$PF_1 + PF_2 = a + \frac{c}{a}x + a - \frac{c}{a}x$$
$$= 2a \qquad \text{............ (6)}$$

Which is true as per our definition of ellipse, so any point that satisfies the equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

lies on the ellipse, thus the equation of an ellipse with centre at origin
and major axis along the x-axis is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

5. Special cases of ellipse:

From the above equation of the ellipse, it is clear that for every point P (x, y) on the ellipse, we have,

$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2} \leq 1,$$

i.e. $x^2 \leq a^2$ or $-a \leq x \leq a$

Therefore, the ellipse lies between the lines $x = -a$ and $x = a$ and touches these lines.

Similarly above equation can be written as,

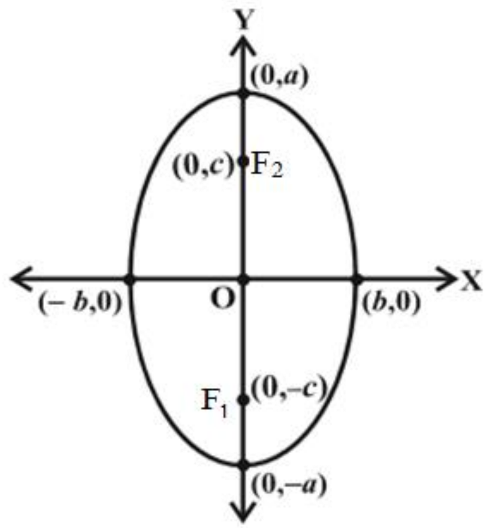
$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \leq 1$$

i.e. $y^2 \le b^2$ or $-b \le x \le b$

Thus, the ellipse lies between the lines $x = -b$ and $x = b$ and touches these lines.

Similarly, we can derive the equation of the ellipse, if its major axis is along y-axis, i.e., if the foci of the ellipse are on y-axis,

The ellipse with major axis along y-axis is shown below,



Proceeding as above, we get the equation of the ellipse as,

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad \ldots\ldots\ldots\ldots\ldots (7)$$

Equations (2) and (7) are as standard equations of the ellipses.

Note:

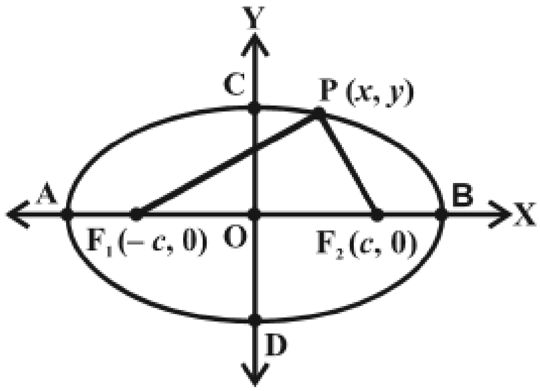
The standard equations of ellipses have centre at the origin and the major and minor axes along coordinate axes. In other words, the foci of the ellipse are on one of the coordinate axis.

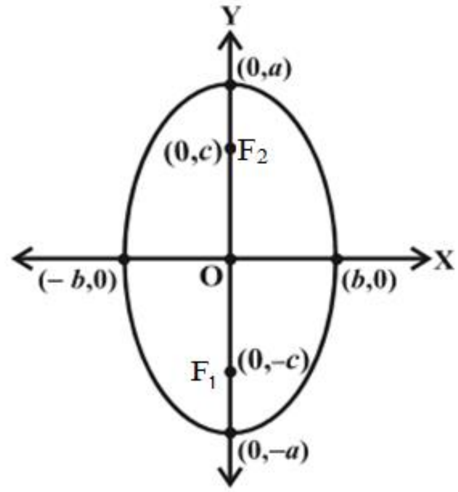
Observations:

From the standard equations of the ellipses, we have the following observations:

1) Ellipse is symmetric with respect to both the coordinate axes,

if (x, y) is a point on the ellipse, then $(-x, y)$, $(x, -y)$ and $(-x, -y)$ are also points on the ellipse.





2) The foci always lie on the major axis. The major axis is along x-axis if the coefficient of x^2 has larger denominator and it is along the y-axis if the coefficient of y^2 has larger denominator.

Example:

Ellipse,

$$\frac{x^2}{36}+\frac{y^2}{16}=1$$

has major axis along x-axis, because x^2 has larger denominator and

ellipse,

$$\frac{x^2}{4}+\frac{y^2}{25}=1$$

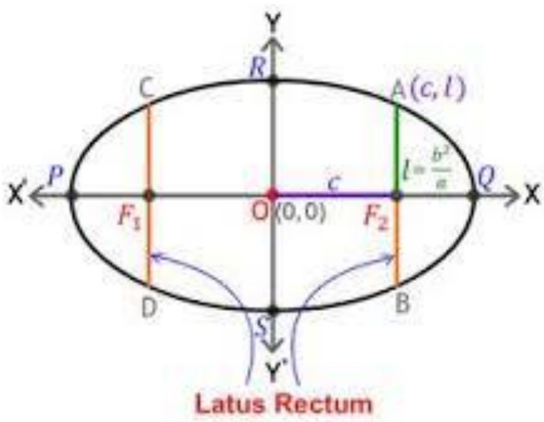
has major axis along y-axis, because denominator of y^2 is greater.

6. Latus rectum:

Latus rectum of an ellipse is a line segment perpendicular to the major axis passing through any of the foci and whose end points lie on the ellipse.

To find the length of the latus rectum of the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$



As per the definition given above, line-segments AB and CD are latus rectum of the ellipse, let us find their lengths.

We know that ellipse is symmetrical about both the coordinate axes, therefore,

Length of latus rectum CD = Length of latus rectum AB

= $2(AF_2)$

Let the length of line segment AF_2 be *l*, since point A lies on the ellipse, therefore, coordinates of point A will be (c, *l*). We have seen earlier that 'c = ae', hence, coordinates of point A will be (ae, *l*),

Equation of the ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

Hence, we get,

$$\frac{(ae)^2}{a^2} + \frac{l^2}{b^2} = 1$$

Solving we get,

$$l^2 = b^2\ (1 - e^2)$$

But $$e^2 = \frac{c^2}{a^2} = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2}$$

Therefore $$l^2 = \frac{b^4}{a^2}, \quad \text{i.e.,} \quad l = \frac{b^2}{a}$$

Ellipse is symmetric with respect to y-axis,

Therefore, $AF_2 = F_2B$

So length of the latus rectum = $2l$

$$= \frac{2b^2}{a}$$

7. Examples

Examples

Find the equation of the ellipse, the ends of whose major axis are

(± 3, 0) and the ends of whose minor axis are (0, ±2).

Solution:

From the question it is clear that major axis of the ellipse is along *x*-axis, let the required equation of the ellipse be,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad (\text{ where } a^2 > b^2)$$

The end points of major axis are (± a, 0) and the end points of minor axis are (0, ± b), comparing with the question we get,

$a = 3$ and $b = 2$,

therefore, required equation of the ellipse is, $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Example:

Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse.

$$\frac{x^2}{49} + \frac{y^2}{36} = 1$$

Solution:

Since denominator of $\frac{x^2}{49}$ is larger than the denominator of $\frac{y^2}{36}$, the major axis is along the x-axis. Comparing the given equation with equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

We get, $a^2 = 49$, $b^2 = 36$, therefore, $a = 7$ and $b = 6$,

$$c = \sqrt{a^2 - b^2} = \sqrt{49 - 36} = \sqrt{13}$$

Coordinates of the foci are: $(-c, 0)$ and $(c, 0)$,

i.e., $(-\sqrt{13}, 0)$ and $(-\sqrt{13}, 0)$

Coordinates of the vertices are: $(-a, 0)$ and $(a, 0)$,

i.e., $(-7, 0)$ and $(7, 0)$

Length of the major axis = $2a = 14$,

Length of the minor axis = $2b = 12$,

Eccentricity = $\frac{c}{a} = \frac{\sqrt{13}}{7}$

Latus rectum = $\frac{2b^2}{a} = \frac{2\times 36}{7}$

$= \frac{72}{7}$

Example:

Find equation of the ellipse whose vertices are (0, ± 13) and foci are

(0, ± 5).

Solution:

Since the vertices are on y-axis, the equation will be of the form

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$ where a is the semi-major axis,

Therefore, $a = 13$,

Coordinate of foci are (0, ± 5), therefore, $c = 5$,

We know that, $c^2 = a^2 - b^2$,

Hence, $25 = 169 - b^2$,

i.e., $b^2 = 144$ or $b = 12$

Substituting the values, the required equation of the ellipse is,

$$\frac{x^2}{144}+\frac{y^2}{169}=1$$

Example:

Find the equation of the ellipse, whose length of the minor axis is 16 and foci are (0, ± 6).

Solution:

Since the foci are on y-axis, the major axis is along the y-axis. So, equation of the ellipse is of the form,

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$ where a is the semi-major axis,

Length of the minor axis is given equal to16,

$\therefore$ $b = \frac{16}{2} = 8$,

Foci are (0, ± 6), therefore, $c = 6$,

We know that, $c^2 = a^2 - b^2$,

Hence, $36 = a^2 - 64$ or $a^2 = 100$

Therefore, the equation of the ellipse is,

$$\frac{x^2}{64}+\frac{y^2}{100}=1$$

Example:

Find equation of the ellipse whose centre is at (0, 0), the length of minor axis is 4 and which passes through the point (2, 1).

Solution:

Since centre of the ellipse is at (0, 0), let the equation of the required ellipse be,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

given that the length of minor axis is 4, therefore, b = 2,

the required ellipse passes through the point (2, 1), therefore

from equation (i), we have,

$$\frac{2^2}{a^2} + \frac{1^2}{2^2} = 1,$$

Solving we get, $a = \frac{4}{\sqrt{3}}$,

Substituting the values in equation (1), the required equation of the ellipse is,

$$\frac{3x^2}{16} + \frac{y^2}{4} = 1$$

Example:

Find equation of the ellipse with centre at (0, 0), the major axis along y-axis and which passes through the points (3, 2) and (1, 6).

Solution:

Since major axis of the ellipse is along y-axis, the equation will be of the form

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$$

(where a2 > b2)

Since, the ellipse passes through the points (3, 2) and (1, 6),

These coordinates will satisfy equation (i), hence we get,

$$\frac{3^2}{b^2} + \frac{2^2}{a^2} = 1 \quad \text{or} \quad \frac{9}{b^2} + \frac{4}{a^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots..(ii)$$

and $$\frac{1^2}{b^2} + \frac{6^2}{a^2} = 1 \quad \text{or} \quad \frac{1}{b^2} + \frac{36}{a^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots.(iii)$$

solving (i) and (ii), we get,

$\frac{1}{a^2} = \frac{1}{40}$ and $\frac{1}{b^2} = \frac{1}{10}$, or a2 = 40 and b2 = 10,

Hence, the required equation of the ellipse is,

$$\frac{x^2}{10} + \frac{y^2}{40} = 1$$

8. Summary:

1. An ellipse is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant.
2. Ellipse is a conic section, obtained from a cone, when it is cut by a plane making an angle β with its axis, such that, $\alpha < \beta < 90^0$, where, α is the semi vertical angle of the cone.
3. Ellipse is the locus of a point which moves in a plane such that its distance from a fixed point in the plane bears a constant ratio to its distance from a fixed straight line in the same plane and the ratio is less than one. This ratio is the eccentricity of the ellipse.
4. The eccentricity of an ellipse is also the ratio between the distances from the centre of the ellipse to one of the foci and to one of the vertices of the ellipse.
5. The equation of an ellipse with foci on the *x*-axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

6. The equation of an ellipse with foci on the y-axis is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

7. Latus rectum of an ellipse is a line segment perpendicular to the major axis through any of the foci, whose end points lie on the ellipse.
8. Length of the latus rectum of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

is $\frac{2b^2}{a}$.