## 1. Details of Module and its structure

| Module Detail |  |
| :--- | :--- |
| Subject Name | Mathematics |
| Course Name | Conic Sections - Part 1 |
| Module Name/Title | kemh_21101 |
| Module I | After going through this lesson, the learners will be able to do the <br> following: <br> • |
| Pre-requisites | - Section of a Cone <br> Objectives |
|  | Section of a Cone, Ellipse, Parabola <br> Sections |
| Keywords |  |

## 2. Development Team

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## 1. Introduction:

Conic Sections, as the name suggests, we are going to study sections cut-off from a cone. Let us see some curves and shapes, which we quite often observe in our surroundings, the most common is this water spring coming out from the tap,


Try to observe the path of the water flow, it's a curve. Similar curve is obtained, when a baseball is thrown up or bomb is shot from a cannon.


Another curve which we quite often find in our surroundings is elliptical and many times circular,


Observe the following object, it is commonly used in our houses to sit upon, the shape of the curve is hyperbola,


These curves are known as conic sections, Apollonius (262 B.C. -190 B.C.) discovered that these curves can be obtained as intersections of a plane with a double napped right circular cone and hence they are known as conic sections. In the following sections we will see how the intersection of a plane with a double napped right circular cone results in different types of curves. These curves have a very wide range of applications in fields such as planetary motion, design of telescopes and antennas, reflectors in flashlights and automobile headlights, etc.

## 2. Section of a Cone:

Let us first understand what we actually mean by double napped right circular cone.
Let $l$ be a fixed vertical line and m be another line intersecting it at a fixed point V and inclined to it at an angle $\alpha$.


Suppose we rotate the line $m$ around the line $l$ in such a way that the angle $\alpha$ remains constant. Then the surface generated is a double-napped right circular hollow cone, extending indefinitely in both the directions as shown below,


The point of intersection of lines $l$ and $m$ is V , it is called the vertex; the line $l$ is known as the axis of the cone. The rotating line $m$ is called generator of the cone.

The vertex separates the cone into two parts called nappes. If we take the intersection of a plane with this hollow cone, the section so obtained is a curve and is known as conic section.

Thus, conic sections are the curves obtained by intersecting a right circular double cone by a plane. We obtain different kinds of conic sections depending on the position of the intersecting plane with respect to the cone and by the angle made by the plane with the vertical axis of the cone. The intersection of the plane with the cone can take place either at the vertex of the cone or at any other part of the nappe either below or above the vertex.
3. Circle, ellipse, parabola and hyperbola: When the plane cuts the nappe (other than the vertex) of the cone, we get circle, ellipse, parabola or hyperbola depending upon the position of the intersecting plane with respect to the cone.

Let $\beta$ be the angle made by the intersecting plane with the vertical axis of the cone, then

i) The section is a circle if $\beta=90^{\circ}$, as is shown in figure below,


$$
\beta=90^{\circ}
$$

ii) When $\alpha<\beta<90^{\circ}$, the section is an ellipse,

iii) When $\alpha=\beta$; the section is a parabola,

(In each of the above three situations, the plane cuts entirely across one nappe of the cone).
iv) When $0 \leq \beta<\alpha$; and the plane does not pass through the vertex, then it will cut both the nappes and the curves of intersection will have two branches known as hyperbola;


## 4. Degenerated conic sections:

When the plane passes through the vertex of double napped cone, again three different cases may arise;
(i)

When $\alpha<\beta \leq 90^{\circ}$, then the section is a point,

(ii) When $\alpha=\beta$, the section is a straight line and it is the generator of the cone. It is the degenerated case of a parabola.

(iii) When $0 \leq \beta<\alpha$, the section is a pair of intersecting straight Lines.

It is the degenerated case of a hyperbola.


Plane passing through axis of the double cone
Let us now define each of these conic sections on the basis of their geometric properties and obtain their equations in standard form.

## 5. Circle:

You have already learnt in earlier classes that;

Circle is the set of all those points in a plane which are equidistant from a fixed point in the plane.

Or
Circle is the path of a moving point which moves in a plane such that it is always at a constant distance from a fixed point.


The fixed point is called the centre of the circle and the constant distance from centre to a point on the circle is known as the radius of the circle.

## Equation of circle:

The equation of the circle is simplest when centre of the circle is at the origin. Let us derive the equation of the circle when centre and radius of the circle are given and then derive the equation for the circle whose centre is at origin.

Let C be the centre of the circle and its coordinates be $(h, k)$ and the radius of the circle be r . Let $\mathrm{P}(\mathrm{x}, \mathrm{y})$ be any point on the circle as shown in the figure below,

Then, by the definition, $\mathrm{CP}=\mathrm{r}$ (radius).


By the distance formula, we have

$$
\sqrt{(x-h)^{2}+(y-k)^{2}}=r
$$

Squaring both the sides we have,

$$
(x-h)^{2}+(y-k)^{2}=r^{2}
$$

This is the required equation of the circle with centre at $(h, k)$ and radius is given equal to r .
To find the equation of a circle with centre at origin $(0,0)$ and radius $r$, we will have to put,
$h=k=0, \quad$ because coordinates of centre are $(0,0)$
Thus, the equation of the circle with centre at origin $(0,0)$ and radius $r$, will be,

$$
x^{2}+y^{2}=r^{2}
$$

## 6. Example:

Find the equation of a circle whose center is at $(2,-4)$ and radius 5.

## Solution:

The equation of the circle with centre at $(h, k)$ and radius r is,

$$
(x-h)^{2}+(y-k)^{2}=r^{2}
$$

Here centre is at $(2,-4)$ and radius is 5 ,

$$
\therefore \quad h=2, k=-4 \text { and } r=5 \text {, }
$$

Substituting in above equation we get,

$$
(x-2)^{2}+(y-(-4))^{2}=5^{2}
$$

Hence the required equation is,

$$
(x-2)^{2}+(y+4)^{2}=25
$$

## Example:

Convert the following equation of circle into standard form. Also find the coordinates of centre of the circle and its radius.

## Solution:

The given equation of the circle is

It can be written as,

$$
\left(x^{2}+10 x\right)+\left(y^{2}-6 y\right)+30=0
$$

completing the squares within the parenthesis, we get

$$
\begin{gathered}
\left(x^{2}+10 x+25\right)+\left(y^{2}-6 y+9\right)=-30+25+9 \\
\Rightarrow \quad(\mathrm{x}+5)^{2}+(\mathrm{y}-3)^{2}=4 \\
\Rightarrow \quad\left(\mathrm{x}-(-5)^{2}+(\mathrm{y}-3)^{2}=2^{2}\right.
\end{gathered}
$$

Therefore, the given circle has centre at $(-5,3)$ and radius is 2 .

## Example:

Find the equation of the circle whose center is at the point $(-3,-6)$ and which passes through the point (-4, 8).

## Solution:

We know that distance between centre of a circle and a point on the circle is equal to the radius of the circle,

Hence radius of the circle $r$ is,

$$
\begin{aligned}
r & =\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}} \\
& =\sqrt{(-3-(-4))^{2}+(-6-8)^{2}} \\
& =\sqrt{(1)^{2}+(-14)^{2}}=\sqrt{1+196} \\
& =\sqrt{197}
\end{aligned}
$$

Now centre of the circle is at the point $(-3,-6)$, therefore required equation of the circle is,

$$
(x+3)^{2}+(y+6)^{2}=197
$$

## Example:

Is the point $P(-3,8)$ inside, outside or on the circle, whose equation is given by $x^{2}+4 x+y^{2}-8 y$ $=5$

## Solution:

Let us write given equation of the circle in standard form, it is

$$
(x+2)^{2}+(y-4)^{2}=5^{2}
$$

Hence center and radius of the circle are,
Center $(-2,4)$ and radius $=5$
Distance between point $\mathrm{P}(-3,8)$ and the center $(-2,4)$ of the circle is given by the formula,

$$
\begin{aligned}
d & =\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}} \\
d & =\sqrt{(-2-(-3))^{2}+(4-8)^{2}} \\
& =\sqrt{(1)^{2}+(-4)^{2}} \\
& =\sqrt{(1)^{2}+(16)^{2}} \\
& =\sqrt{17} \\
& =4.12 \text { (approx) }<5(\text { radius of the circle })
\end{aligned}
$$

Since the distance of the point P from the center of the circle is smaller than its radius, hence point P is inside the circle.

## Example:

Find the equation of the circle that passes through the points
$A(2,1)$ and $B(-2,3)$ and has its center on the line:
$x+y+4=0$.

## Solution:

Let coordinates of the centre of the circle be (h, k),
Then, equation of the circle will be,

$$
(x-h)^{2}+(y-k)^{2}=r^{2}
$$

Since the circle passes through the points $(2,1)$ and $(-2,3)$, then these points will lie on the circle, hence they will satisfy the equation of the circle, therefore we have,

$$
\begin{equation*}
(2-h)^{2}+(1-k)^{2}=r^{2} \tag{1}
\end{equation*}
$$

and

$$
(-2-h)^{2}+(3-k)^{2}=r^{2}
$$

Also, since the centre of the circle lies on the line

$$
\begin{array}{ll} 
& x+y+4=0 \\
\text { we have, } & h+k+4=0 \tag{3}
\end{array}
$$

Solving the equations (1), (2) and (3),
we get, $\quad h=-2, k=-2$ and $r=5$

Hence, the equation of the required circle is

$$
(x-(-2))^{2}+(y-(-2))^{2}=5^{2}
$$

Solving we get,

$$
x^{2}+y^{2}+4 x+4 y-17=0, \text { this is the required equation of the circle. }
$$

## 7. Summary

1. When we rotate a line around another line in such a way that the angle between them remains constant then the surface generated is a double-napped right circular hollow cone, extending indefinitely in both the directions.
2. The curves that are obtained as intersections of a plane with a double napped right circular cone are known as conic sections.
3. When the plane cuts the nappe (other than the vertex) of the cone, we get circle, ellipse, parabola or hyperbola depending upon the position of the intersecting plane with respect to the cone.
4. A circle is the set of all points in a plane that are equidistant from a fixed point in the plane.
5. The fixed point is called the centre of the circle and the constant distance from centre to a point on the circle is known as the radius of the circle.
6. The equation of a circle with centre $(h, k)$ and the radius $r$ is given by $(x-h)^{2}+(y-k)=$ $r^{2}$.
$\square$
