1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 02 (Class XI, Semester - 2)
Module Name/Title	Some Special Series-Part 5
Module I	kemh_20905
Pre-requisites	Arithmetic Progression, Geometric Progression
Objectives	After going through this lesson, the learners will be able to do the following: • Evaluate sum to n terms of first n natural numbers • Evaluate sum to n terms of square of first n natural numbers • Evaluate sum to n terms of cubes of first n natural numbers • Use these formulae of sums to solve the questions
Keywords	Series, nth term, Sum to n terms, squares of natural numbers, cubes of natural numbers

2. Development Team

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Table of Contents :

1. **Introduction**

Till now, we are very well aware of the terms sequences, arithmetic progressions, geometric progression and series. We have also learnt to find general term and sum to n terms of A.P. and G.P. We have also learnt to evaluate sum to n terms of infinite geometric series. So now what is new? We have already learnt to find sum of finite and infinite progressions which are nothing but to find the sum of series.

We know to evaluate the sum of this series : 1 + 2 + 3 + 4 + ….

We know this is A.P. and sum can be evaluated by using formula

$S_n = \frac{n}{2}[2a + (n-1)d]$

Right. But do we know to evaluate sum to n terms of the series as follows:

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots$$

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots$$

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \cdots$$

This module deals in evaluating sum to n terms of the series of this kind. Let's go through this module and at the end we will be able to evaluate sum to n terms of all such special series.

2. **Series**

We have already learnt a little bit about series in module 1. Let's recall what is a series.

Let $a_1, a_2, a_3, a_4, a_5, \ldots\ldots a_n$ be any sequence the expression $a_1 + a_2 + a_3 + a_4 + a_5 + \ldots + a_n$ is nothing but the series corresponding to the sequence. Sum of the series means the number obtained when all the terms of the series are being added up. For convenience sake, series is denoted by Greek letter *Sigma* as $\sum_{k=1}^{n} a_k$ which means summation of $a_1, a_2, a_3, a_4, a_5, \ldots\ldots a_n$

Thus, it can be observed that the series represent the sum of terms but not the sum itself.

Another difference lies in the representation that terms of the sequence are separated by commas and terms of the series are separated by the + sign.

Here in this module, we will learn about determining n^{th} or general term of the special series and then evaluating its sum to n terms. These special series comprises of sum of n terms of first n natural numbers, sum of their squares and sum of their cubes.

3. Sum of First n Natural Numbers

Consider the series 1 + 2 + 3 + .. + n

We know that this is an A.P. with first term a = 1 and common difference d = 1 and

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$\frac{n}{2}(a + l)$$

$$\frac{n}{2}(n + 1)$$

$\frac{n(n+1)}{2}$

Or we can say that $\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots = \frac{n(n+1)}{2}$

4. Sum of Squares of First n Natural Numbers

Consider the series $1^2 + 2^2 + 3^2 + 4^2 + \cdots$

Consider the identity $(x+1)^3 - x^3 = 3x^2 + 3x + 1$

Putting x = 1, 2, 3, …, n – 1, n successively, we get

$$2^3 - 1^3 = 3.1^2 + 3.1 + 1$$

$$3^3 - 2^3 = 3.2^2 + 3.2 + 1$$

$$4^3 - 3^3 = 3.3^2 + 3.3 + 1$$

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$$n^3 - (n-1)^3 = 3.(n-1)^2 + 3.(n-1) + 1$$

$$(n+1)^3 - n^3 = 3.n^2 + 3.n + 1$$

Adding column wise,

$$(n+1)^3 - 1^3 = 3[1^2 + 2^2 + 3^2 + \cdots + n^2] + 3[1 + 2 + \cdots + n] + [1 + 1 + \cdots + 1]$$

$$3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + n$$

$$\Rightarrow n^3 + 3n^2 + 3n = 3\sum_{k=1}^{n} k^2 + 3.\frac{n(n+1)}{2} + n$$

$$\Rightarrow 3\sum_{k=1}^{n} k^2 = n^3 + 3n^2 + 3n - 3.\frac{n(n+1)}{2} - n$$

$$\frac{2n^3+6n^2+6n-3n^2-3-2n}{2}$$

$$\frac{2n^3+3n^2+n}{2}$$

$$\frac{n(n+1)(2n+1)}{2}$$

$$\Rightarrow \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$, we can say that \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

5. **Sum of Cubes of First n Natural Numbers**

Consider the series $1^3 + 2^3 + 3^3 + 4^3 + \cdots$

Consider the identity $(x+1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1$

Putting x = 1, 2, 3, …, n – 1, n successively, we get

$$2^4 - 1^4 = 4.1^3 + 6.1^2 + 4.1 + 1$$

$$3^4 - 2^4 = 4.2^3 + 6.2^2 + 4.2 + 1$$

$$4^4 - 3^4 = 4.3^3 + 6.3^2 + 4.3 + 1$$

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$$n^4 - (n-1)^4 = 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 1$$

$$(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1$$

Adding column wise,

$$(n+1)^4 - 1^4 = 4[1^3 + 2^3 + 3^3 + \cdots + n^3] + 6[1^2 + 2^2 + 3^2 + \cdots + n^2] + 4[1 + 2 + 3 + \cdots + n] + n$$

$$\Rightarrow n^4 + 4n^3 + 6n^2 + 4n = 4\sum_{k=1}^{n} k^3 + 6\left[\frac{n(n+1)(2n+1)}{6}\right] + 4\left[\frac{n(n+1)}{2}\right] + n$$

$$k^3 = n^4 + 4n^3 + 6n^2 + 4n - 6\left[\frac{n(n+1)(2n+1)}{6}\right] - 4\left[\frac{n(n+1)}{2}\right] - n$$

$$\Rightarrow 4\sum_{k=1}^{n}$$

$n^4 + 4n^3 + 6n^2 + 4n - n(n+1)(2n+1) - 2n(n+1) - n$

$n^4 + 2n^3 + n^2$

$n^2(n+1)^2$

$$\Rightarrow \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} = \left[\frac{n(n+1)}{2}\right]^2$$

$$, we can say that \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

6. **Sum to n terms of Special Series**

If we have to evaluate sum to n terms of special series like $1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \cdots$ or $4^3 \times 1 + 8^3 \times 2 + 12^3 \times 3 + \cdots$ then we can use the following steps:

- Write n^{th} or general term of the series as $T_n = (a_n of 1st series) \times (a_n of 2nd series)$
- Express T_n as $T_n = an^3 + bn^2 + cn + d$
- Evaluate sum to n terms of the series as

$$S_n = a\sum_{k=1}^{n} k^3 + b\sum_{k=1}^{n} k^2 + c\sum_{k=1}^{n} k + dn$$

Let's check out some examples based on this.

Example 1: Find sum to n terms of the series: $1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots$

Solution: Let T_n and S_n be the general term and Sum to n terms of the series.

$$T_n = (a_n of 1,2,3,\ldots)^2 \times (a_n of 2,3,4 \ldots)$$

$$[1 + (n-1)1]^2 \times [2 + (n-1)1]$$

$$n^2(n+1) = n^3 + n^2$$

$$S_n = \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2$$

$$\left[\frac{n(n+1)}{2}\right]^2+\frac{n(n+1)(2n+1)}{6}$$

$$\frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{2n+1}{3}\right]$$

$$\frac{n(n+1)}{2}\left[\frac{3n(n+1)+2(2n+1)}{6}\right]$$

$$\frac{n(n+1)}{2}\left[\frac{3n^2+3n+4n+2}{6}\right]$$

$$\frac{n(n+1)}{2}\left[\frac{3n^2+7n+2}{6}\right]$$

$$\frac{n(n+1)}{12}(3n^2+6n+n+2)$$

$$\frac{n(n+1)}{12}[3n(n+2)+1(n+2)]$$

$$\frac{n(n+1)(n+2)(3n+1)}{12}$$

Example 2: Find sum to n terms of the series: 4.8 + 5.9+6.10+7.11+…

Solution: Let T_n and S_n be the general term and Sum to n terms of the series.

$T_n=(a_n of 4,5,6 \ldots)\times(a_n of 8,9,10,\ldots)$

$$[4+(n-1)1]\times[8+(n-1)1]$$

$$(n+3)(n+7)$$

$$n^2+10n+21$$

$$S_n=\sum_{k=1}^{n}k^2+10\sum_{k=1}^{n}k+21n$$

$$\frac{n(n+1)(2n+1)}{6}+10\frac{n(n+1)}{2}+21n$$

$$n\left[\frac{(n+1)(2n+1)}{6}+5(n+1)+21\right]$$

$$\frac{n}{6}\big((n+1)(2n+1)+30(n+1)+126\big)$$

$$\frac{n}{6}(2n^2+33n+157)$$

Example 3: Find sum to n terms of the series: 1 + (1 + 2) + (1 + 2 + 3) +…

Solution: Let T_n and S_n be the general term and Sum to n terms of the series.

$T_n=1+2+3+\cdots+n=\sum_{k=1}^{n}k=\frac{n(n+1)}{2}=\frac{1}{2}(n^2+n)$

$$S_n = \frac{1}{2}\sum_{k=1}^{n} k^2 + \frac{1}{2}\sum_{k=1}^{n} k$$

$$\frac{1}{2}\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\frac{n(n+1)}{2}$$

$$\frac{1}{2}\frac{n(n+1)}{2}\left[\frac{2n+1}{3} + 1\right]$$

$$\frac{n(n+1)}{4}\left(\frac{2n+1+3}{3}\right)$$

$$\frac{n(n+1)}{4}\left(\frac{2n+4}{3}\right)$$

$$\frac{2n(n+1)(n+2)}{12}$$

$$\frac{n(n+1)(n+2)}{6}$$

Example 4: Find sum to n terms of the series whose nth term is given by $a_n = n^3 + 5^n$

Solution: $S_n = \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 5^k$

$$\left[\frac{n(n+1)}{2}\right]^2 + [5^1 + 5^2 + 5^3 + \cdots + 5^n]$$

$$\frac{n^2(n+1)^2}{4} + \frac{5(5^n - 1)}{5-1}$$

$$\frac{1}{4}[n^2(n+1)^2 + 5^{n+1} - 5]$$

Example 5: Find sum to n terms of the series:

$1.n + 2.(n-1) + 3.(n-2) + \cdots + (n-1).2 + n.1$

Solution: Let T_k and S_n be the general term and Sum to n terms of the series.

$T_k = k[n - (k-1)] = k(n-k+1) = (n+1)k - k^2$

$$S_n = (n+1)\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2$$

$$(n+1)\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$\frac{n(n+1)}{2}\left[(n+1) - \frac{(2n+1)}{3}\right]$$

$$\frac{n(n+1)}{2}\left[\frac{3n+3-2n-1}{3}\right]$$

$$\frac{n(n+1)(n+2)}{6}$$

7. Sum to n terms of the Series by Method of Difference

Sometimes the series is such as the difference between the successive terms forms an A.P. or G.P. In that case, n^{th} term of the series is calculated by the method of difference and then sum to n terms is evaluated.

Following examples will help in understanding this method.

Example 6: Find sum to n terms of the series: 5 + 7 + 11 + 17 + …

Solution: Difference between successive terms of the series are 7 – 5 = 2 , 11 – 7 = 4 , 17 – 11 = 6 which makes an A.P.

Let T_n and S_n be the general term and Sum to n terms of the series.

$$S_n = 5 + 7 + 11 + 17 + \cdots + T_{n-1} + T_n$$

$$S_n = 5 + 7 + 11 + 17 + \cdots + T_{n-1} + T_n$$

Subtracting,

$$0 = 5 + [2 + 4 + 6 + \cdots + (T_n - T_{n-1})] - T_n$$

$$\Rightarrow T_n = 5 + \left(\frac{n-1}{2}\right)[4 + (n-2)2]$$

$$5 + \frac{(n-1)(2n)}{2}$$

$$5 + n(n-1)$$

$$n^2 - n + 5$$

$$S_n = \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k + 5n$$

$$\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 5n$$

$$\frac{n(n+1)}{2}\left[\frac{2n+1}{3} - 1\right] + 5n$$

$$\frac{n(n+1)(2n-2)}{6} + 5n$$

$$\frac{n(n+1)(n-1)}{3} + 5n$$

$$\frac{n(n^2-1)}{3} + 5n$$

$$\frac{n(n^2-1) + 15n}{3}$$

$$\frac{n^3 - n + 15n}{3}$$
$$\frac{n^3 + 14n}{3}$$
$$\frac{n(n^2 + 14)}{3}$$

Example 7: Find sum to n terms of the series: 1 + 3 + 7 + 15 + 31 + …

Solution: Difference between successive terms of the series are 3 – 1 = 2 , 7 – 3 = 4 , 15 – 7 = 8 , 31 – 15 = 16, which makes an G.P.

Let T_n and S_n be the general term and Sum to n terms of the series.

$$S_n = 1 + 3 + 7 + 15 + 31 + \cdots + T_{n-1} + T_n$$
$$S_n = 1 + 3 + 7 + 15 + 31 + \cdots + T_{n-1} + T_n$$

Subtracting,

$$0 = 1 + [2 + 4 + 8 + \cdots + (T_n - T_{n-1})] - T_n$$
$$\Rightarrow T_n = 1 + \frac{2(2^{n-1} - 1)}{2 - 1} = 1 + 2^n - 2 = 2^n - 1$$
$$S_n = \sum_{k=1}^{n} 2^k - n = [2^1 + 2^2 + 2^3 + \cdots + 2^n] - n$$
$$\frac{2(2^n - 1)}{2 - 1} - n$$
$$2^{n+1} - n - 2$$

8. Sum to n terms of the Series by the Method of Partial Fraction

Sometimes series is given in the form of fractions. In that case we can split the terms in two fraction by using partial fractions and then summing all the terms. This method is illustrated by using following examples. In place of remembering big formulae, try to observe the pattern of the terms and their respective partial fractions.

Example 8: Find the sum to n terms of the series: $\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \cdots$

Solution: Let T_n and S_n be the nth term and sum to n terms of the series respectively.

Here in the series, $T_1 = \frac{1}{3.5} = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)$

$$T_2 = \frac{1}{5.7} = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right)$$

$$T_3 = \frac{1}{7.9} = \frac{1}{2}\left(\frac{1}{7} - \frac{1}{9}\right)$$

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$$T_n = \frac{1}{2}\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)$$

$$S_n = \frac{1}{2}\left[\frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \frac{1}{7} - \frac{1}{9} + \cdots + \frac{1}{2n+1} - \frac{1}{2n+3}\right]$$

$$\frac{1}{2}\left[\frac{1}{3} - \frac{1}{2n+3}\right]$$

$$\frac{1}{2}\left[\frac{2n+3-3}{3(2n+3)}\right]$$

$$\frac{n}{3(2n+3)}$$

Example 9: Find the sum to n terms of the series: $\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \frac{1}{11.14}$ ...

Solution: Let T_n and S_n be the nth term and sum to n terms of the series respectively.

Here in the series, $T_1 = \frac{1}{2.5} = \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5}\right)$

$$T_2 = \frac{1}{5.8} = \frac{1}{3}\left(\frac{1}{5} - \frac{1}{8}\right)$$

$$T_3 = \frac{1}{8.11} = \frac{1}{3}\left(\frac{1}{8} - \frac{1}{11}\right)$$

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$$T_n = \frac{1}{3}\left(\frac{1}{3n-1} - \frac{1}{3n+2}\right)$$

$$S_n = \frac{1}{3}\left[\frac{1}{2} - \frac{1}{5} + \frac{1}{5} - \frac{1}{8} + \frac{1}{8} - \frac{1}{11} + \cdots + \frac{1}{3n-1} - \frac{1}{3n+2}\right]$$

$$\frac{1}{3}\left(\frac{1}{2} - \frac{1}{3n+2}\right)$$

$$\frac{1}{3}\left(\frac{3n+2-2}{2(3n+2)}\right)$$

$$\frac{n}{2(3n+2)}$$

Example 10: Find the sum to n terms of the series: $\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \cdots$

Solution: Let T_n and S_n be the nth term and sum to n terms of the series respectively.

Here in the series, $T_1 = \frac{1}{3.5} = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)$

$$T_2 = \frac{1}{5.7} = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right)$$

$$T_3 = \frac{1}{7.9} = \frac{1}{2}\left(\frac{1}{7} - \frac{1}{9}\right)$$

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$$T_n = \frac{1}{2}\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)$$

$$S_n = \frac{1}{2}\left[\frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \frac{1}{7} - \frac{1}{9} + \cdots + \frac{1}{2n+1} - \frac{1}{2n+3}\right]$$

$$\frac{1}{2}\left[\frac{1}{3} - \frac{1}{2n+3}\right]$$

$$\frac{1}{2}\left[\frac{2n+3-3}{3(2n+3)}\right]$$

$$\frac{1}{2}\left(\frac{2n}{3(2n+3)}\right)$$

$$\frac{n}{3(2n+3)}$$

9. Summary

- Let a_1, a_2, a_3, a_4, a_5, ……a_n be any sequence the expression $a_1 + a_2 + a_3 + a_4 + a_5 + \ldots + a_n$ is nothing but the series corresponding to the sequence.
- Sum of the series means the number obtained when all the terms of the series are being added up.
- Series is denoted by Greek letter *Sigma* as $\sum_{k=1}^{n} a_k$ which means summation of a_1, a_2, a_3, a_4, a_5, ……a_n
- $\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots = \frac{n(n+1)}{2}$
- $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$