1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 02 (Class XI, Semester - 2)
Module Name/Title	Arithmetic Mean & Geometric Mean-Part 4
Module I	kemh_20904
Pre-requisites	Arithmetic Progression, Geometric Progression
Objectives	After going through this lesson, the learners will be able to do the following: • Calculate Arithmetic Mean • Calculate Geometric Mean • Determine the relationship between arithmetic and geometric mean
Keywords	Arithmetic Mean, Geometric Mean

2. Development Team

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1. Introduction

Let's consider two numbers 5 and 20. Can we calculate their average? Yes, it's pretty easy. We add 5 and 20 and divide by 2 and get 12.5 as arithmetic mean or average. Now if we are interested in finding their geometric mean then how to do? Geometric mean! What is that? Never heard about it. Let's challenge ourselves once more. Which four numbers can be added between 5 and 20 to make it an A.P.? No idea!

Go through this module to get answers to all such questions.

Till now, we are well aware of arithmetic and geometric progressions. We have learnt to determine their n^{th} terms and sum to n terms. We are well versed with their applications to real life. Now we need to know their theoretical applications like arithmetic mean and geometric mean, which we are going to study in this module.

2. Arithmetic Mean (A.M.)

Let a, b and c be an A.P. Then number b is called arithmetic mean (A.M.) of a and b if

b – a = c – b

or, 2b = a + c

or, $b = \frac{a+c}{2}$

If we have to insert on arithmetic mean between any two numbers a and c then we can certainly use this formula and we must ensure that a, b and c are in A.P. Let's understand this by one example.

Example 1: Insert one A.M. between 20 and 22

Solution: Let a be A.M. to be inserted between 20 and 22 then we can say that

$$a = \frac{20 + 22}{2} = \frac{42}{2} = 21$$

We can easily observe that 20, 21 and 22 are in A.P.

Now what will happen if more than one A.M.'s are to be inserted between 20 and 22?

Example 2: Insert two A.M. between 20 and 22.

Solution: Let A_1 and A_2 be two arithmetic means between 20 and 22 such that 20, A_1 , A_2, 22 is an A.P.

a = 20

$a_4 = 22 \Longrightarrow a + 3d = 22 \Longrightarrow 3d = 22 - 20 = 2 \Longrightarrow d = \frac{2}{3}$

$$A_1 = a + d = 20 + \frac{2}{3} = \frac{62}{3}$$

$$A_2 = a + 2d = 20 + \frac{4}{3} = \frac{64}{3}$$

20, $\frac{62}{3}$, $\frac{64}{3}$, 22 is an A.P.

It does not seem to be an A.P. Check out for common difference. Don't want to calculate?

Use equivalent fraction and write A.P. again

$\frac{60}{3}, \frac{62}{3}, \frac{64}{3}, \frac{66}{3}$

Now it is pretty clear that this is an A.P. and two A.M. between 20 and 22 are $\frac{62}{3}$, $\frac{64}{3}$

Now we have learnt to insert one and two A.M. between any two numbers a and b. It is time to generalize the method and define A.M.

Let a and b be any two numbers then $A_1, A_2, \dots, A_n$ are 'n' A.M.'s between a and b if $a, A_1, A_2, \dots, A_n, b$ is an A.P.

Let's take some more examples on A.M.

Example 3: Insert 4 A.M. between 7 and 27.

Solution: Let $A_1, A_2, A_3 \wedge A_4$ be 4 A.M.'s between 7 and 27.

Then $7, A_1, A_2, A_3, A_4, 27$ is an A.P.

Here, a = 7 and a_6 = 27

$$a_6 = a + 5d \Rightarrow 27 = 7 + 5d \Rightarrow 5d = 20 \Rightarrow d = 4$$

A_1= a + d = 7 + 4 = 11

A_2 = a + 2d = 7 + 8 = 15

A_3 = a + 3d = 7 + 12 = 19

A_4 = a + 4d = 7 + 16 = 23

Thus, 7, 11, 15, 19, 23, 27 is an A.P.

Example 4: Insert A.M.'s between 6 and 50 in such a way that 4^{th} A.M. is 22. Find the number of A.M.s inserted.

Solution: Let $A_1, A_2, \ldots, A_n$ be n numbers to be inserted between 6 and 50 in such a way that $6, A_1, A_2, \ldots, A_n, 50$ is an A.P.

Here, a = 6 and

$a_{n+2} = 50$

$$\Rightarrow a + (n+1)d = 50$$

$$\Rightarrow 6 + (n+1)d = 50$$

$\Rightarrow (n+1)d = 44$ ……………(1)

$$Also given that A_4 = 22$$

$$\Rightarrow a_5 = 22$$

$$\Rightarrow a + 4d = 22$$

$$\Rightarrow 4d = 22 - 6 = 16$$

$$\Rightarrow d = 4$$

$$Using value of d \in (1), we have$$

$$(n+1)4 = 44$$

$$\Rightarrow n + 1 = 11$$

$$\Rightarrow n = 10$$

Example 5: If n numbers are to be inserted between 10 and 37 so that the resulting sequence is an A.P. If ratio of second and last number is 8:17 then find the value of n.

Solution: Let $A_1, A_2, \ldots, A_n$ be n numbers to be inserted between 10 and 37 such that $10, A_1, A_2, \ldots, A_n, 37$ is an A.P.

a = 10

$$a_{n+2} = 37$$

$$\Rightarrow a + (n+1)d = 37$$

$$\Rightarrow 10 + (n+1)d = 37 \Rightarrow d = \frac{27}{n+1}$$

$$Given that A_2 : A_n = 8 : 17$$

$$\Rightarrow \frac{a_3}{a_{n+1}} = \frac{8}{17}$$

$$\Rightarrow \frac{a+2d}{a+nd} = \frac{8}{17}$$

$$\Rightarrow \frac{10+2\left(\frac{27}{n+1}\right)}{10+n\left(\frac{27}{n+1}\right)} = \frac{8}{17}$$

$$\Rightarrow \frac{10(n+1)+54}{10(n+1)+27n} = \frac{8}{17}$$

$$\Rightarrow \frac{10n+64}{37n+10} = \frac{8}{17}$$

$$\Rightarrow 170n + 1088 = 296n + 80$$

$$\Rightarrow 126n = 1008$$

$$\Rightarrow n = \frac{1008}{126}$$

$$\Rightarrow n = 8$$

3. **Geometric Mean (G.M.)**

Now what is this new term, Geometric mean? Mean; that also geometric? Sounds absurd? No it is not, actually. We have learnt about Geometric Progressions. Let three numbers a, b and c are in G.P. then we know that

$$\frac{b}{a} = \frac{c}{b} \Rightarrow b^2 = ac \Rightarrow b = \sqrt{ac}$$

This can be termed as ‘b’ is Geometric Mean (G.M.) of a and c.

That sounds great! Isn’t it?

Let’s take one example to do a little brainstorming.

Example 6: Insert a G.M. between two numbers 3 and 27.

Solution: Let G be a G.M. to be inserted between 3 and 27 then 3, G, 27 is a G.P.

G = $\sqrt{3 \times 27} = \sqrt{81} = 9$

Thus 3, 9, 27 is a G.P.

Example 7: Insert two G.M. between 7 and 448.

Solution: Let G_1 and G_2 be two G.M. such that 7, G_1, G_2, 448 is a G.P.

Here, a = 7 and

$$a_4 = 448$$

$$\Rightarrow ar^3 = 448$$

$$\Rightarrow r^3 = \frac{448}{a} = \frac{448}{7} = 64$$

$$\Rightarrow r = 4$$

$$G_1 = a_2 = a.r = 7 \times 4 = 28$$

$$G_2 = a_3 = a.r^2 = 7 \times 16 = 112$$

Thus two G.M.s are 28 and 112 and resulting G.P. is 7, 28, 112, 448

That was the case for insertion of one or two G.M.s what will happen if we have to insert more G.M.s

Before studying that, let's define G.M.

Let a and b be any two numbers then numbers $G_1, G_2, G_3, \dots, G_n$ are n Geometric Means (G.M.) between a and b if $a, G_1, G_2, G_3, \dots, G_n, b$ is a G.P.

Example 8: Insert 5 G.M. between $\frac{1}{64}$ and 64

Solution: Let G_1, G_2, G_3, G_4, G_5 be five G.M. to be inserted so that $\frac{1}{64}, G_1, G_2, G_3, G_4, G_5, 64$ is a G.P.

Here, $a = \frac{1}{64}$

and $a_7 = 64$

$$\Rightarrow a.r^6 = 64$$

$$\Rightarrow r^6 = \frac{64}{a} = \frac{64}{\frac{1}{64}}$$

$$\Rightarrow r^6 = 64 \times 64 = 4^6$$

$$\Rightarrow r = 4$$

$$G_1 = a_2 = a.r = \frac{1}{64} \times 4 = \frac{1}{16}$$

$$G_2 = a_3 = a.r^2 = \frac{1}{64} \times 16 = \frac{1}{4}$$

$$G_3 = a_4 = a.r^3 = \frac{1}{64} \times 64 = 1$$

$$G_4 = a_5 = a.r^4 = \frac{1}{64} \times 256 = 4$$

$$G_5 = a_6 = a.r^5 = \frac{1}{64} \times 1024 = 16$$

Thus five G.M. between $\frac{1}{64}$ and 64 are $\frac{1}{16}$, $\frac{1}{4}$, 1, 4 and 64

Example 9: Let 'n' numbers have been inserted between $\sqrt{3}$ and 81, find the value of n and r.

Solution: Let $G_1, G_2, G_3, \dots, G_n$ be n G.M. to be inserted so that $\sqrt{3}, G_1, G_2, G_3, \dots, G_n, 81$ is a G.P.

Here, $a_1 = a = \sqrt{3} \wedge a_{n+2} = 81$

$$\Rightarrow a_{n+2} = a.r^{n+1} = 81$$

$$\Rightarrow r^{n+1} = \frac{81}{a} = \frac{81}{\sqrt{3}} = \frac{\left(\sqrt{3}\right)^8}{\sqrt{3}} = \left(\sqrt{3}\right)^7$$

$$\Rightarrow r = \sqrt{3} \wedge n + 1 = 7 \Rightarrow n = 6$$

Example 10: n GMs $G_1, G_2, G_3, \dots, G_n$ have been inserted between a and b such that ratio between G_2 and G_n is 1 : 625. If a = 2 then find the values of n and b.

Solution: Given that $2, G_1, G_2, G_3, \dots, G_n, b$ is a G.P.

Also given that $\frac{G_2}{G_n} = \frac{1}{625}$

$$\Rightarrow \frac{a_3}{a_{n+1}} = \frac{1}{625}$$

$$\Rightarrow \frac{ar^2}{ar^n} = \frac{1}{625}$$

$$\Rightarrow \frac{1}{r^{n-2}} = \frac{1}{625}$$

$$\Rightarrow r^{n-2} = 625 = 5^4$$

$$\Rightarrow r = 5 \wedge n - 2 = 4 \Rightarrow n = 6$$

$$Also b = a_{n+2} = ar^{n+1} = 2 \times 5^7 = 156250$$

4. Relationship between A.M. and G.M.

We have studied about A.M. and G.M. of two numbers. Let's denote A.M. and G.M. of any two numbers by A and G. Are A and G related to each other somehow? Let's study some properties of A and G and solve some problems based on A and G.

Property I: For any two positive numbers a and b, their arithmetic mean is always larger than the geometric mean, i.e., A > G

Proof: We know $A = \frac{a+b}{2} \wedge G = \sqrt{ab}$

$$\therefore A - G = \frac{a+b}{2} - \sqrt{ab}$$

$\frac{a+b-2\sqrt{ab}}{2}$

$\frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 > 0$

$$\Rightarrow A - G > 0 \Rightarrow A > G$$

Property II: If A and G are arithmetic and geometric mean of any two numbers a and b then the quadratic equation whose roots are a and b is given by $x^2 - 2Ax + G^2 = 0$

Proof: We know $A = \frac{a+b}{2} \wedge G = \sqrt{ab}$

If a and b are roots of a quadratic equation then equation is

$$x^2 - (a+b)x + ab = 0$$

$$\Longrightarrow x^2 - 2Ax + G^2 = 0$$

Property III: If A and G be the A.M. and G.M. of two positive numbers a and b then the numbers are given by $A \pm \sqrt{A^2 - G^2}$

Proof: We know that the equation using A and G is

$x^2 - 2Ax + G^2 = 0$

$$\Longrightarrow x = \frac{2A \pm \sqrt{4A^2 - 4G^2}}{2}$$

$$\Longrightarrow x = \frac{2A \pm 2\sqrt{A^2 - G^2}}{2}$$

$$\Longrightarrow x = A \pm \sqrt{A^2 - G^2}$$

$$Thus roots of the quadratic equation are A \pm \sqrt{A^2 - G^2}$$

Example 11: Write the quadratic equation whose roots are a and b and AM and GM of a and b are 60 and 12 respectively.

Solution: Quadratic equation is $x^2 - 2Ax + G^2 = 0 \Longrightarrow x^2 - 120x + 144 = 0$

Example 12: Find two positive numbers a and b whose sum is 20 and whose AM exceeds GM by 2.

Solution: Let A and G denote AM and GM of a and b

Given that a + b = 20 $\Longrightarrow \frac{a+b}{2} = 10 \Longrightarrow A = 10$

A – G = 2

$$\Longrightarrow G = A - 2 = 8$$

$$\Longrightarrow \sqrt{ab} = 8$$

$$\Longrightarrow ab = 64$$

$$We know, (a-b)^2 = (a+b)^2 - 4ab$$

$400 - 256$

144

$$\Longrightarrow a - b = 12$$

$$Solvingequations, a + b = 20 \wedge a - b = 12,$$

$$wegeta = 16 \wedge b = 4$$

Example 13: Find two positive numbers a and b such that their sum is 650 and sum of their AM and GM is 450.

Solution: Let A and G denote AM and GM of a and b then

$a + b = 650 \Longrightarrow \frac{a+b}{2} = 325 \Longrightarrow A = 325$

$$A + G = 450 \Longrightarrow G = 450 - A = 450 - 325 = 125$$

$$weknowa \wedge baregivenbyA \pm \sqrt{A^2 - G^2} = A \pm \sqrt{(A+G)(A-G)}$$

$$325 \pm \sqrt{(450)(200)}$$

$$325 \pm \sqrt{4 \times 9 \times 25 \times 100}$$

$$325 \pm 2 \times 3 \times 5 \times 10$$

$$325 \pm 300$$

$$625,25$$

Hence a = 625 and b = 25

Example 14: Let a and b be two positive numbers and A and G be their AM and GM respectively. Find a and b if their difference is 240, product is 4096 and difference between A and G is 72.

Solution: Given that a – b = 240 , a x b = 4096 and A – G = 72

$$G = \sqrt{ab} = \sqrt{4096} = 64$$

A – G = 72 ⟹ A = 72 + G = 72 + 64 = 136

$$\Longrightarrow \frac{a+b}{2} = 136 \Longrightarrow a + b = 272$$

$$Solvingequations, a + b = 272 \wedge a - b = 240,$$

a = 16 and b = 256

Example 15: Find two positive numbers a and b whose sum is 90 and ratio of their AM and GM is 5 : 3.

Solution: Given that a + b = 90 ⟹ $\frac{a+b}{2} = 45 \Longrightarrow AM = 45$

AM : GM = 5 : 3

$$\Rightarrow \frac{\frac{a+b}{2}}{\sqrt{ab}} = \frac{5}{3}$$

$$\Rightarrow \frac{45}{\sqrt{ab}} = \frac{5}{3}$$

$$\Rightarrow \sqrt{ab} = 45 \times \frac{3}{5} = 27$$

$$\Rightarrow ab = 729$$

$$We know, (a-b)^2 = (a+b)^2 - 4ab$$

$90^2 - 4 \times 729$

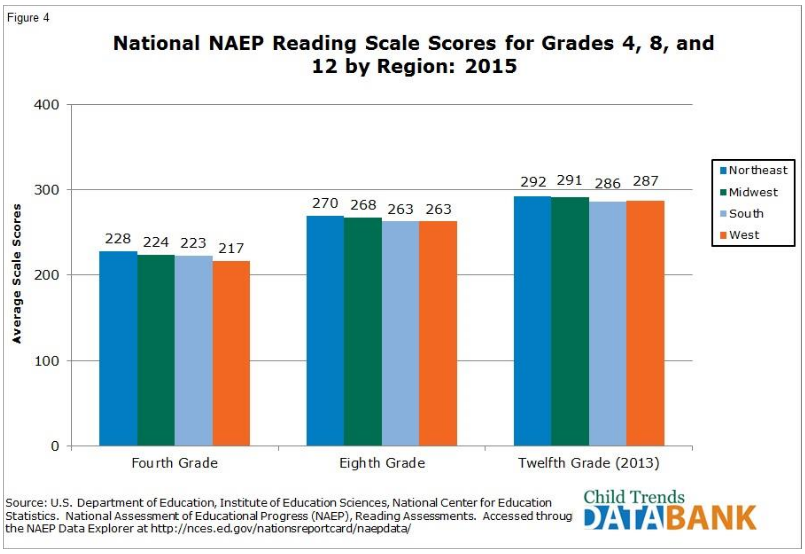
$8100 - 2916$

$5184 = 72^2$

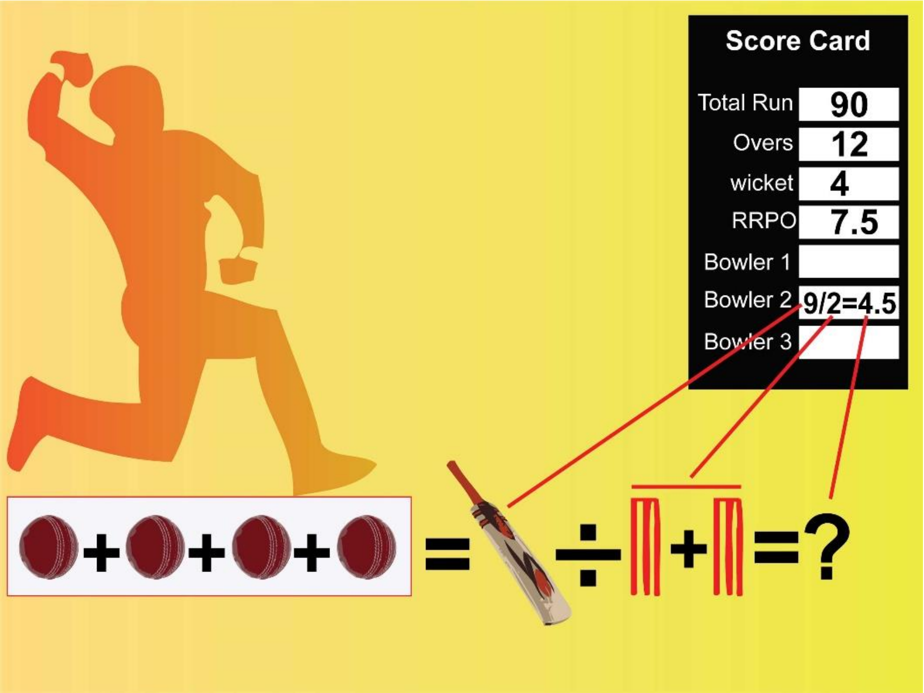
Solving, $a - b = 72 \wedge a + b = 90$, we get a = 81 and b = 9

5. **Real Life Applications**

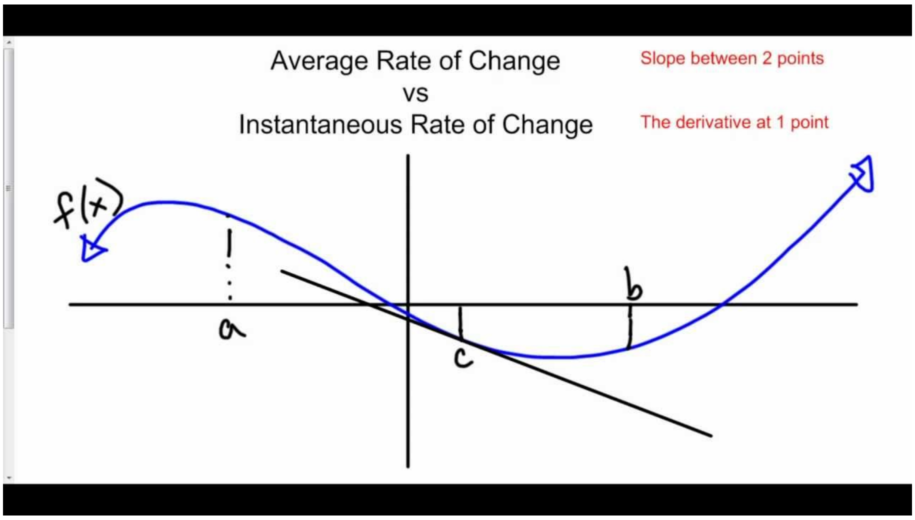
- A.M. is used to find mean score of a set of students.



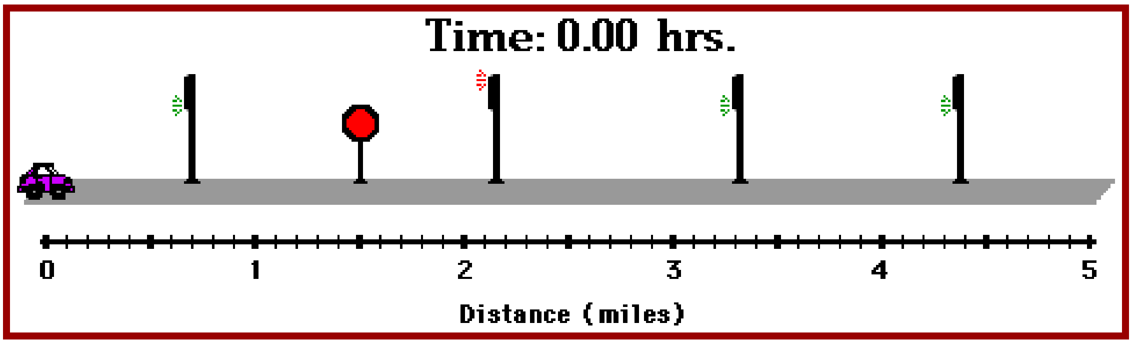
- Average run rate for a batsman in the cricket match.



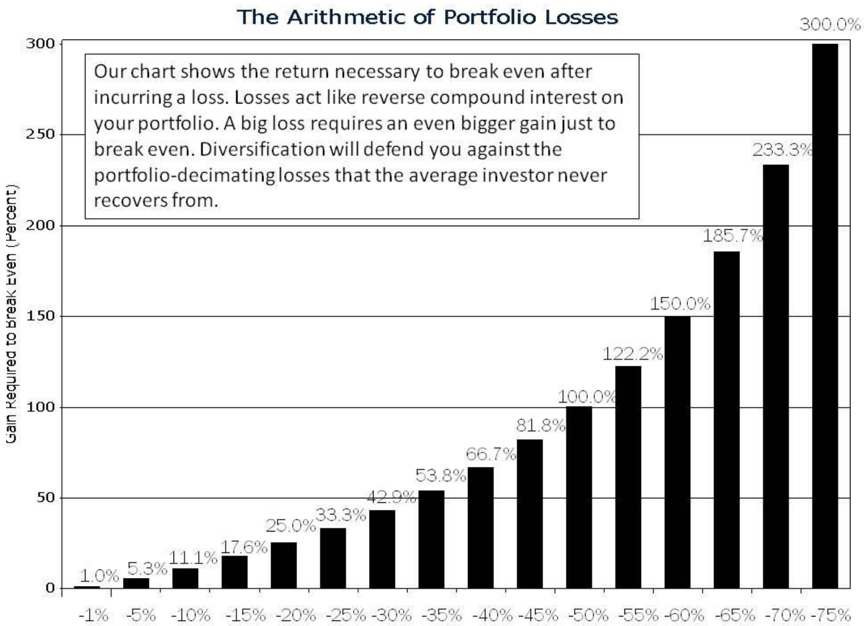
- Average rate of change of any quantity



- Average speed of any vehicle



- A.M. is used to calculate arithmetic returns of investment



- G.M. is used to calculate annual investment returns or compound returns

Summary

- Let a and b be any two numbers then $A_1, A_2, \ldots, A_n$ are 'n' A.M.'s between a and b if $a, A_1, A_2, \ldots, A_n, b$ is an A.P.
- Number b is called arithmetic mean (A.M.) of a and b if $b = \frac{a+c}{2}$
- Let a and b be any two numbers then numbers $G_1, G_2, G_3, \ldots, G_n$ are n Geometric Means (G.M.) between a and b if $a, G_1, G_2, G_3, \ldots, G_n, b$ is a G.P.
- 'b' is Geometric Mean (G.M.) of a and c if $b = \sqrt{ac}$

- For any two positive numbers a and b, their arithmetic mean is always larger than the geometric mean, i.e., A > G
- If A and G are arithmetic and geometric mean of any two numbers a and b then the quadratic equation whose roots are a and b is given by $x^2 - 2Ax + G^2 = 0$
- If A and G be the A.M. and G.M. of two positive numbers a and b then the numbers are given by $A \pm \sqrt{A^2 - G^2}$