## 1. Details of Module and its structure

| Module Detail | |
|---|---|
| Subject Name | Mathematics |
| Course Name | Mathematics 02 (Class XI, Semester - 2) |
| Module Name/Title | Geometric Progression (G.P.)-Part 3 |
| Module I | kemh_20903 |
| Pre-requisites | Patterns, Sequence, Arithmetic Progression |
| Objectives | After going through this lesson, the learners will be able to do the following:<br>• Identify Geometric progression (G.P.)<br>• Determine general term of G.P.<br>• Determine sum to n terms of G.P.<br>• Calculate sum of infinite G.P.<br>• Apply G.P. to solve real life problems |
| Keywords | G.P., Common Ratio, General term, Sum to n terms, infinite G.P. |

## 2. Development Team

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**Table of Contents :**



## 1. Introduction

You must have heard about the story of wheat and chess. Don't remember? Let's just revisit our childhood and try to recall the story. It has many versions.

There was a wealthy king. He called for a game and announced that winner will get the prize of this choice. One man who won the game, asked the king to give him grains of wheat to be filled his chess. King laughed at him for such a foolish and meagre prize. But the winner set a condition for the king that king had to give him one grain of wheat for the First Square of chess and then keep on doubling the grains till $64^{th}$ square is reached. King nodded in full agreement.

After some time, the treasurer reported to the king that this might take the entire grains of the kingdom. But the king was dumbfounded that how is this possible. Do you know the answer? The number of grains made a sequence like 1, 2, 4, 8, 16, 32, 64,…. till 64 terms. When all these numbers were added then the sum was such a huge number that it could have taken entire wheat of the kingdom. How is this possible? We will discuss the answer in same module later.

## 2. Geometric Progression (G.P.)

Till now, you are very much familiar with A.P. – Arithmetic Progression. If a constant quantity is added in or subtracted from all the terms of a sequence then it is called arithmetic progression. In the same manner, if a constant non-zero quantity multiplies or divides all the terms then it is called a geometric progression.

e.g. 2, 4, 6, 8, 10, …. is an A.P. because here constant difference d = 2 which is being added to all the terms to get the next term. 2, 4, 8, 16, 32,…. is a G.P. because 2 is being multiplied to all the terms to get the subsequent term.

We can define G.P. as follows:

A sequence $a_{1,}a_2, a_3, \dots.., a_n, \dots.$ is called a geometric progression if each term is non-zero and $\frac{a_{k+1}}{a_k} = r(constant)fork \geq 1$

For a G.P., 'a' is the first term and 'r' is the common ratio.

We can also write a G.P. as $a, ar, ar^2, ar^3, ar^4, \dots..$ where '$a$' is first term and $r \neq 0$ is the common ratio of G.P.

**Example 1**: Identify the type of sequence:

(i) 2, 3, 45, 7, 11, 13, ….

(ii) 2, 4, 6, 8, 10, 12,….

(iii) 2, 4, 8, 16, 32, 64,…..

**Solution**: (i) is a sequence of prime numbers

(ii) This is an A.P. as d = 2

(iii) This is a G.P. as r = 2

**Example 2**: Is given sequence a G.P.? If yes, find the common ratio. $8,4,2,1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots \dots$

**Solution:** $\frac{a_2}{a_1} = \frac{4}{8} = \frac{1}{2}, \frac{a_3}{a_2} = \frac{8}{16} = \frac{1}{2}, \frac{a_4}{a_3} = \frac{1}{2}$

Since, $\frac{a_{n+1}}{a_n} = \frac{1}{2} = number$, therefore, given sequence is a G.P. with r = 1/2

## 3. General Term or n$^{th}$ term of G.P.

Consider a G.P. $\sqrt{3}, 3, 3\sqrt{3}, 9, 9\sqrt{3}, \dots..$

Here, you can observe that $a_1 = \sqrt{3} = 3^{\frac{1}{2}}$

$$r = \frac{3}{\sqrt{3}} = \sqrt{3}$$

Also, we can rewrite the sequence as $a_1 = \sqrt{3}, a_2 = \sqrt{3}(\sqrt{3})^1$, $a_3 = \sqrt{3}(\sqrt{3})^2$,$a_4 = \sqrt{3}(\sqrt{3})^3$ and so on

This shows that power of common ratio ‘r’ is one less than the number of term, i.e., if number of term is 1 then power of ‘r’ is 0; if number of term is 2 then power of ‘r’ is 1; if number of term is 3 then power of ‘r’ is 2 and so on. If this process continues then we can write the general term or $n^{th}$ term of G.P. as

$$a_n = \sqrt{3}\left(\sqrt{3}\right)^{n-1}$$

That means the formula of general term of G.P. can be generalized as $a_n = a.r^{n-1}$

**Example 3:** Find $10^{th}$ term of G.P. 1, 5, 25, 125,…..

**Solution:** Here, a = 1, r = 5

$$a_n = a.r^{n-1} = 1 \times 5^{n-1} \Rightarrow a_{10} = 1 \times 5^9 = 5^9$$

**Example 4**: 4096 is which number term of G.P. $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1,2,4, \dots ..$

**Solution**: Here, a = 1/8, r = 2

$$a_n = a.r^{n-1}$$

$$\Rightarrow 4096 = \frac{1}{8}2^{n-1}$$

$$\Rightarrow 4096 \times 8 = 2^{n-1}$$

$$\Rightarrow 2^{n-1} = 2^{12} \times 2^3 = 2^{15}$$

$$\Rightarrow n - 1 = 15$$

$$\Rightarrow n = 16$$

**Example 5**: In a G.P. if $6^{th}$ term is 25 times the $4^{th}$ term then find value of common ratio.

**Solution**: $a_6 = 25a_4 \Rightarrow a.r^5 = 25a.r^3 \Rightarrow r^2 = 25 \Rightarrow r = \pm 5$

4. **Selection of Terms of G.P.**

Sometimes for the sake of convenience of calculations, specially when product of terms is given, terms of G.P. are taken as follows:

| No. of Terms | Terms of G.P. | Common Ratio |
|---|---|---|
| 3 | $\frac{a}{r}, a, ar$ | r |
| 4 | $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$ | $r^2$ |
| 5 | $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$ | r |

If the product of the terms is not given, then the numbers are taken as $a, ar, ar^2, ar^3, ar^4, \dots ..$

**Example 6:** Find common ratio of G.P. consisting of three terms whose sum is 147/4 and product is 343.

**Solution:** Let three terms of G.P. be $\frac{a}{r}, a, ar$

Given that $\frac{a}{r} \times a \times ar = 343$

$$\Rightarrow a^3 = 343 \Rightarrow a = 7$$

$$Also, \frac{a}{r} + a + ar = \frac{147}{4}$$

$$\Rightarrow a\left(\frac{1}{r} + 1 + r\right) = \frac{147}{4}$$

$$\Rightarrow 7\left(\frac{1}{r} + 1 + r\right) = \frac{147}{4}$$

$$\Rightarrow \left(\frac{1}{r} + 1 + r\right) = \frac{21}{4}$$

$$\Rightarrow \frac{1 + r + r^2}{r} = \frac{21}{4}$$

$$\Rightarrow 4r^2 + 4r - 17 = 0$$

$$\Rightarrow 4r^2 - 16r - r + 4 = 0$$

$$\Rightarrow 4r(r - 4) - 1(r - 4) = 0$$

$$\Rightarrow (4r - 1)(r - 4) = 0$$

$$\Rightarrow r = 4, \frac{1}{4}$$

5. **Sum to n terms**

We need to evaluate sum of n terms of G.P. or sum of n terms of geometric series

Let $S_n = a + ar + ar^2 + ar^3 + \cdots + ar^n$

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n + ar^{n+1}$$

$$Subtracting, rS_n - S_n = ar^{n+1} - a$$

$$\Rightarrow S_n(r - 1) = a(r^{n+1} - 1)$$

$$\Rightarrow S_n = \frac{a(r^{n+1} - 1)}{(r - 1)}, r \neq 1$$

Here we say that $r \neq 1$. What will happen if r = 1?

If r = 1 then $S_n$ = a + a + a + a +….+ a = na

By using this formula of $S_n$, we can fond total of any number of terms of geometric series.

Remember the story in the beginning about grains of wheat and chess. Let's revisit the story once. If king has to give double the number of grains of wheat to the winner for every square of the chess than the previous square then it becomes a G.P. as follows:

$$2^0 + 2^1 + 2^2 + 2^4 + \cdots + 2^{63}$$

$Totalnumberofwheatgrains = S_{64} = \frac{2(2^{63}-1)}{2-1}$=18,446,744,073,709,551,615 grains of wheat, weighing 461,168,602,000 metric tons, which would be a heap of wheat larger than Mount Everest.

**Example 7**: find the sum to n terms of the series: $\frac{1}{9}, \frac{1}{3}, 1, ...$

**Solution**: $a = \frac{1}{9}, r = 3$

$$S_7 = \frac{\frac{1}{9}(3^7 - 1)}{3 - 1} = \frac{1}{9} \times \frac{1}{2} \times (2187 - 1) = \frac{1093}{9}$$

**Example 8**: Find the sum to n terms of the series: 3 + 33 + 333+…

**Solution**: $S_n = 3 + 33 + 333 + \cdots$

$$3(1 + 11 + 111 + \cdots)$$

$$\frac{3}{9}(9 + 99 + 999 + ..)$$

$$\frac{3}{9}[(10 - 1) + (10^2 - 1) + (10^3 - 1) + \cdots]$$

$$\frac{3}{9}[(10 + 10^2 + 10^3 + \cdots) - n]$$

$$\frac{3}{9}\left[\frac{10(10^n - 1)}{10 - 1} - n\right]$$

$$\frac{3}{81}(10^{n+1} - 9n - 10)$$

**Example 9:** Find n if a = 2, r = 3 and Sn = 6560

**Solution:** $S_n = 6560 \Rightarrow \frac{2(3^n-1)}{3-1} = 6560 \Rightarrow 3^n - 1 = 6560 \Rightarrow 3^n = 6561 = 3^8 \Rightarrow n = 8$

**Example 10**: The ratio of the first three terms is to that of first six terms of G.P. is 64:189. Find the common ratio.

**Solution**: $\frac{S_3}{S_6} = \frac{64}{189}$

$$\Rightarrow \frac{\frac{a(r^3-1)}{r-1}}{\frac{a(r^6-1)}{r-1}} = \frac{64}{189}$$

$$\Rightarrow \frac{r^3-1}{r^6-1} = \frac{64}{189}$$

$$\Rightarrow \frac{r^3-1}{(r^3-1)(r^3+1)} = \frac{64}{189}$$

$$\Rightarrow 189 = 64(r^3+1)$$

$$\Rightarrow 64r^3 = 125$$

$$\Rightarrow r^3 = \frac{125}{64}$$

$$\Rightarrow r = \frac{5}{4}$$

6. **Properties of G.P.**

- If all the terms of G.P. be multiplied or divided by the same non-zero constant then the G.P. remains same with same common ratio.
- e.g. For the G.P. 2, 4, 8, 16,… if all the terms are multiplied by 2 the resulting sequence is 4, 8, 16, 32,… which is also a G.P. with the same common ratio 2.
- The reciprocals of a G.P. form a G.P. with common ratio as 1/r, where r is common ratio of the given G.P.
- e.g. the sequence of reciprocals of G.P. 2, 4, 8, 16,…is ½, ¼, 1/8, 1/16,… which is also a G.P.
- If all the terms of the G.P. are raised to the same power then resulting sequence is also a G.P.
- e.g. if terms of G.P. 2, 4, 8, 16,… are raised to power to 2 then the resulting sequence 4, 16, 64,.. is also a G.P. with common ratio $r^2$

- In a finite G.P., the product of the terms equidistant from the beginning and the end is always same and is equal to the product of the first and the last term.
- Three non-zero numbers a, b and c are in G.P. iff $b^2 = ac$
- Let 3, 6, 12 is a G.P. then $6^2 = 3x12$ i.e., 36 = 36 or vice-versa

7. **Infinite G.P.**

The infinite G.P. is a G.P. whose last term is not fixed. That means, the G.P. $a_{1,}a_2, a_3, \dots\dots, a_n, \dots.$ is called infinite G.P. Till now, we were calculating sum of n terms of finite G.P.; for finding sum of infinite G.P. we can consider one example:

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots.$$

a = 1, r = ½ , r < 1

$$S_n = \frac{1\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = 2\left(1-\left(\frac{1}{2}\right)^n\right)$$

As n is increasing, $\left(\frac{1}{2}\right)^n$ is decreasing and approaching to zero.

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

$$Asn \to \infty, r^n \to 0 \wedge alsosince|r| < 1$$

$$S_\infty = S = \frac{a}{1-r}$$

If r > 1 then S is very large.

**Example 11**: Find sum to infinity of the G.P. $1, \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{4}, \dots$

**Solution:** $S_\infty = \frac{1}{1-\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1}$

**Example 12**: Evaluate: $9^1 . 9^{\frac{1}{2}} . 9^{\frac{1}{4}} \dots$ till infinity

**Solution**: $9^1 . 9^{\frac{1}{2}} . 9^{\frac{1}{4}} \dots = 9^{1+\frac{1}{2}+\frac{1}{4}+\dots+\infty} = 9^{\frac{1}{1-\frac{1}{2}}} = 9^2 = 81$

**Example 13**: Find sum to infinity of the series: $\left(\frac{1}{3}+\frac{1}{7}\right)+\left(\frac{1}{9}+\frac{1}{14}\right)+\left(\frac{1}{27}+\frac{1}{28}\right)+\cdots$

**Solution**: $\left(\frac{1}{3}+\frac{1}{7}\right)+\left(\frac{1}{9}+\frac{1}{14}\right)+\left(\frac{1}{27}+\frac{1}{28}\right)+\cdots$

$$\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\right)+\left(\frac{1}{7}+\frac{1}{14}+\frac{1}{28}+\cdots\right)$$

$$\frac{\frac{1}{3}}{1-\frac{1}{3}}+\frac{\frac{1}{7}}{1-\frac{1}{2}}$$

$$\frac{\frac{1}{3}}{\frac{2}{3}}+\frac{\frac{1}{7}}{\frac{1}{2}}$$

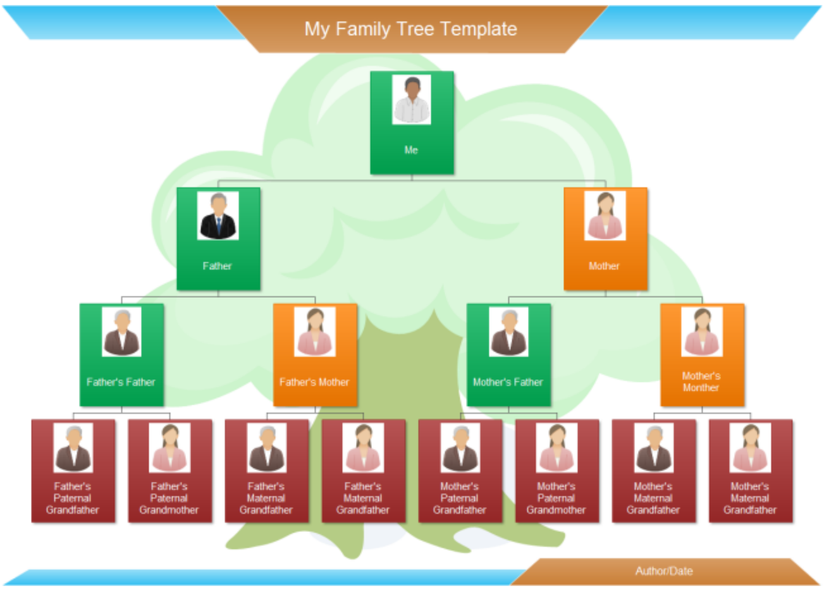
$$\frac{1}{2}+\frac{2}{7}$$

$$\frac{7+4}{14}$$

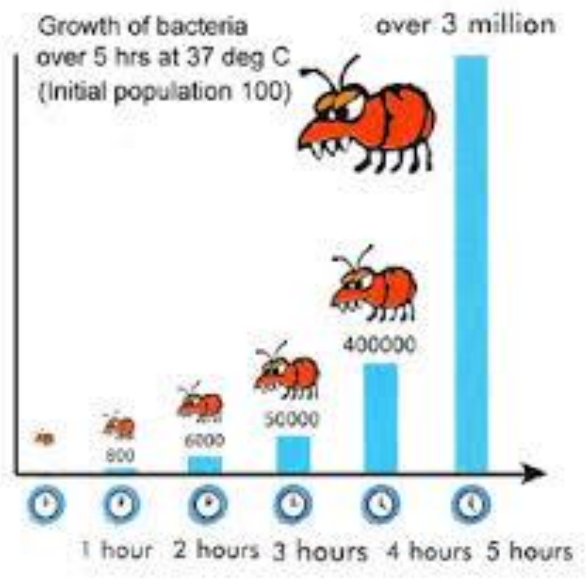
$$\frac{11}{14}$$

8. **Real Life Applications of G.M.**

- The number of ancestors or decedents in any family tree. Two parent will have 2 children; who will further give rise to two sets of parents and so on.

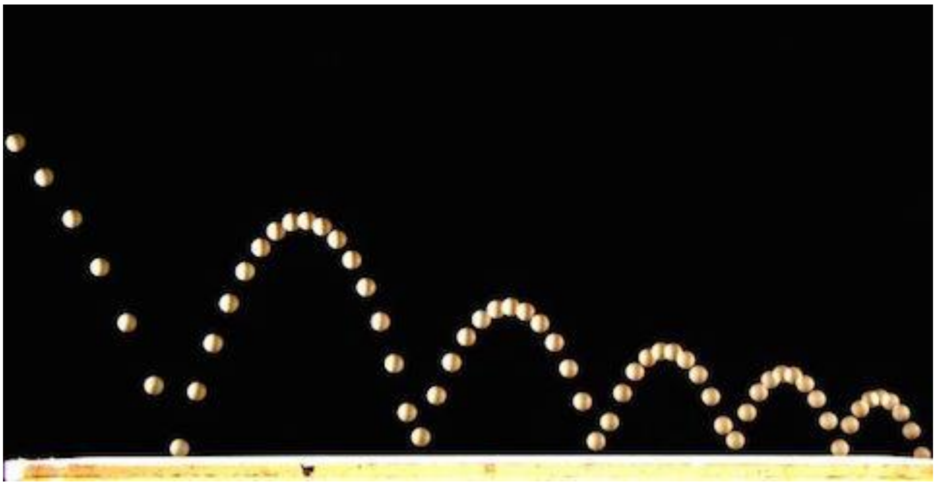



- The number of bacteria in any culture grows exponentially which forms a G.P.

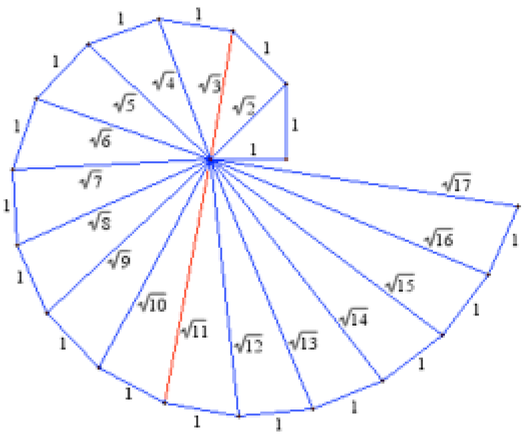



- Compound interest gives the amount which makes G.P.


- The height of the successive bounce of the tennis ball or any ball when it jumps follows G.P.



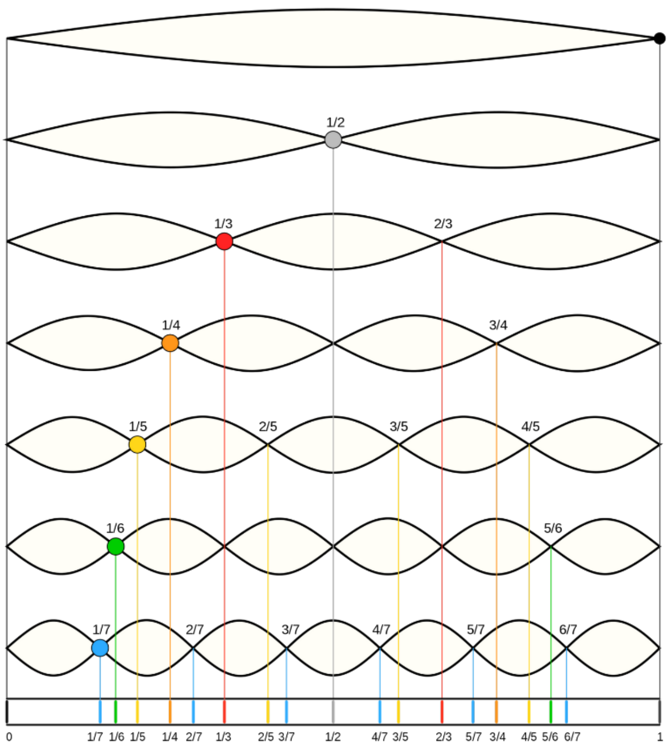
- Square-root spiral- The lengths of each of segment make G.P.



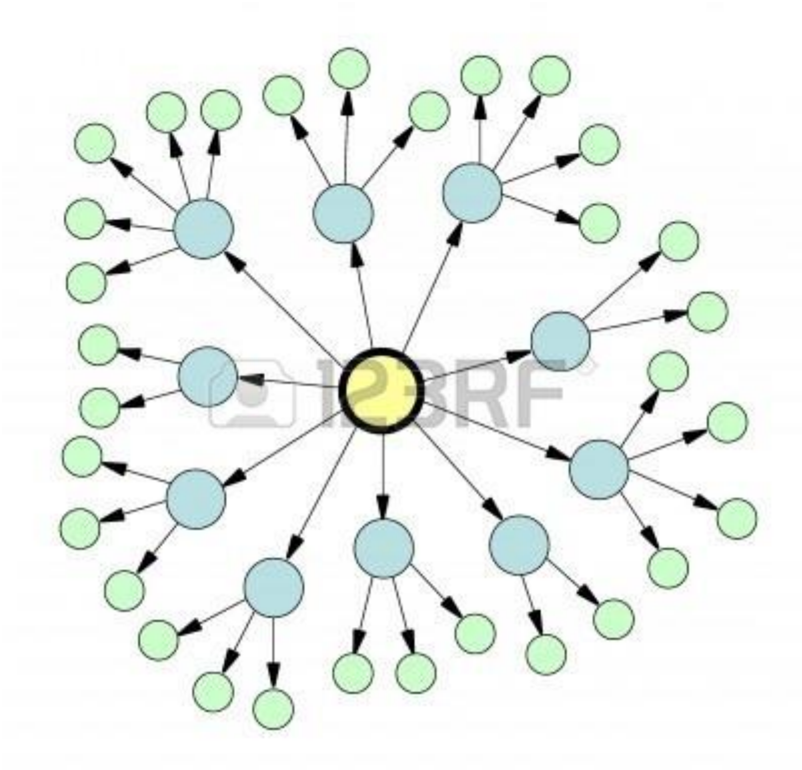

- Series of repeating decimal follows a G.P.

| | |
|---|---|
| $0.\dot{2}$ | 0.22222222222222... |
| $0.1\dot{2}$ | 0.12222222222222... |
| $0.\dot{1}2\dot{3}$ | 0.12312312312312 3... |
| $0.1\dot{2}\dot{3}$ | 0.12323232323232 3... |

- Resonating musical harmonics follows G.P.

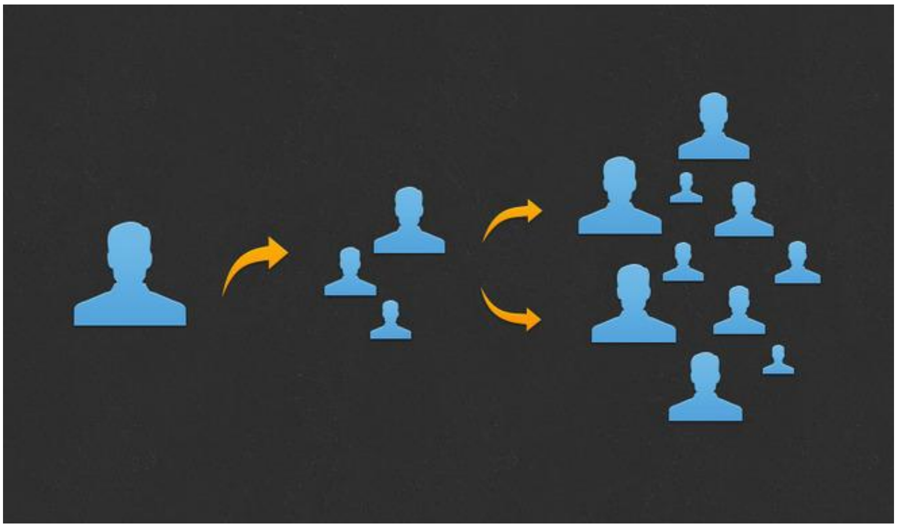


- Social networking- one person is connected to 4 people, who further are connected to 4 who further connected to 4 and so on.

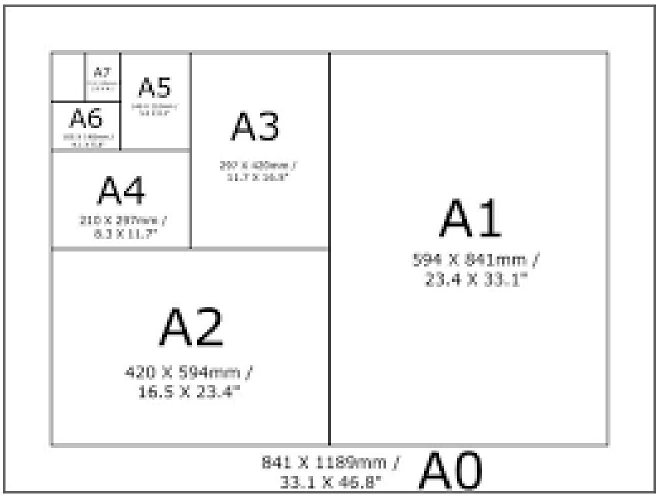


- Multi-level marketing or pyramid selling or network marketing follows G.P.

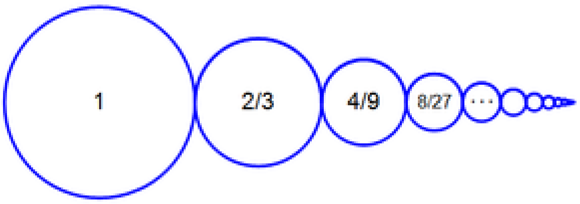

- Product distribution also follows G.P.



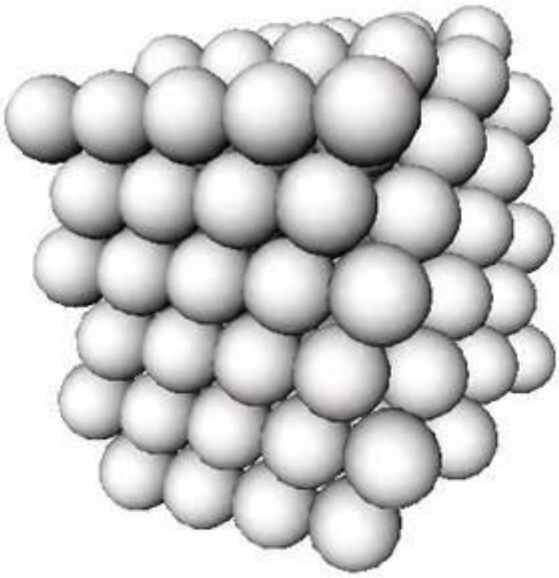
- Paper Sizes A0, A1, A2, A3 etc. follows G.P.



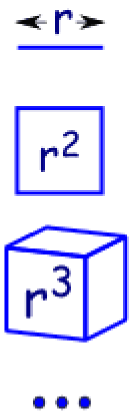

- Similar figures follows G.P.



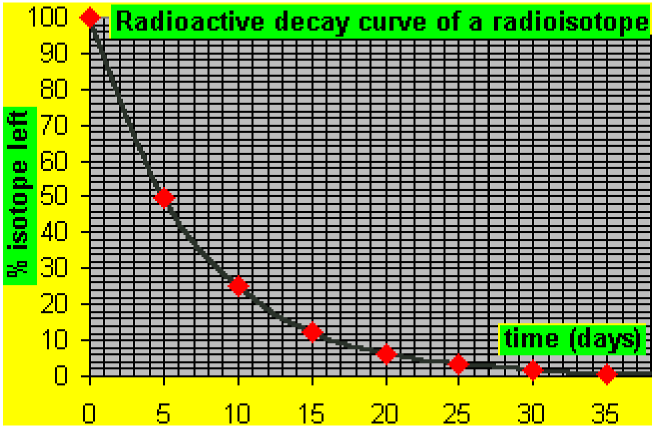

- Hexagonal closed packing of equal spheres – one sphere is covered with three spheres which are further covered or joined by three more spheres and so on.



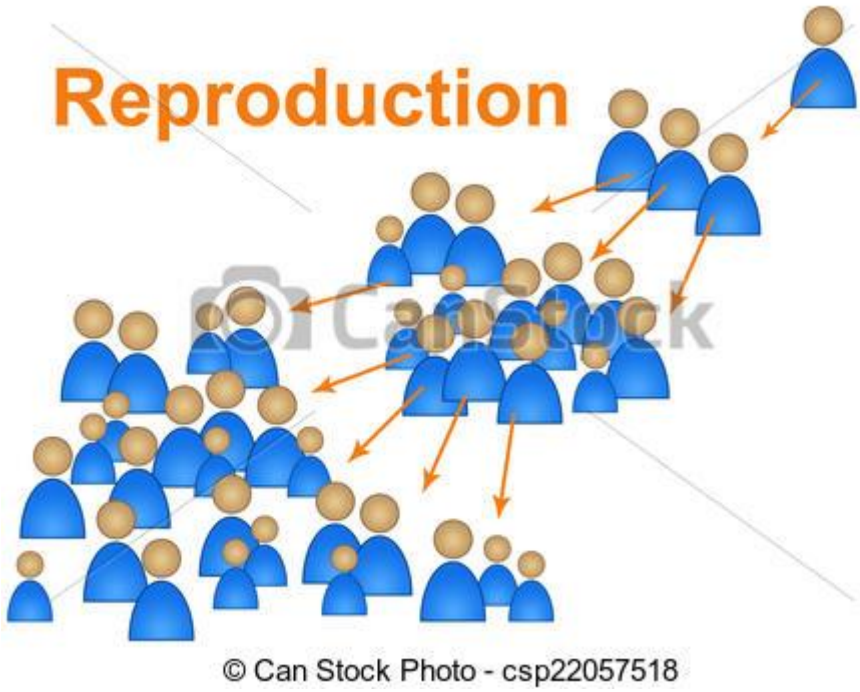
- G.P. is like increasing dimensions in the geometry. A line of length ‘r’ units is 1-dimensional, a square of area ‘$r^2$’ units is 2-dimensional and a cube of volume ‘$r^3$’ is 3-dimensional and so on.



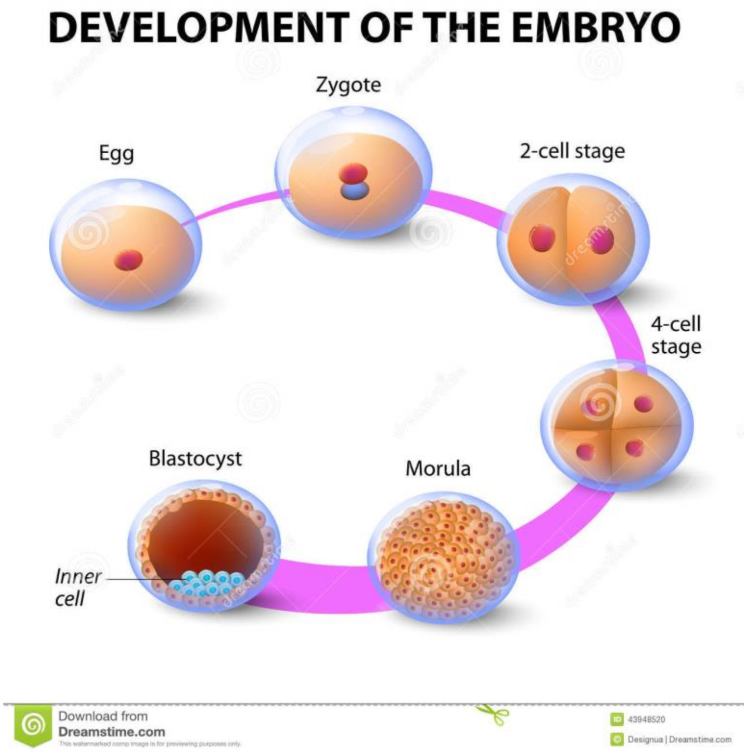

- Given the rate of travel, it is possible to apply concept of G.P. to determine the number of miles a vehicle travels in a given amount of time, and to calculate the distance at any time along the trip.
- Physicists use geometric progressions to calculate the amount of radioactive material left after any given number of half-lives of the material. During each half-life, the material decays by 50 percent.



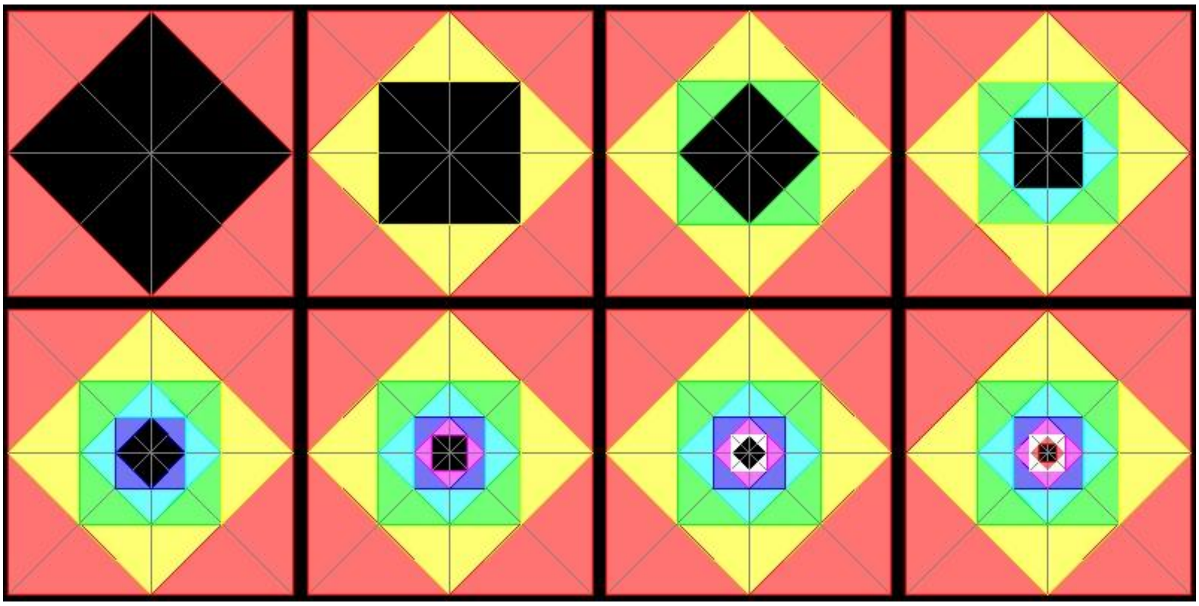

- Population Growth – two people have 3 children who will further have 3 more children and so on.




- Embryo Development – embryo starts with one cell which further gives rise to more cells exponentially.




- Construction of triangles, squares, rectangles by paper folding

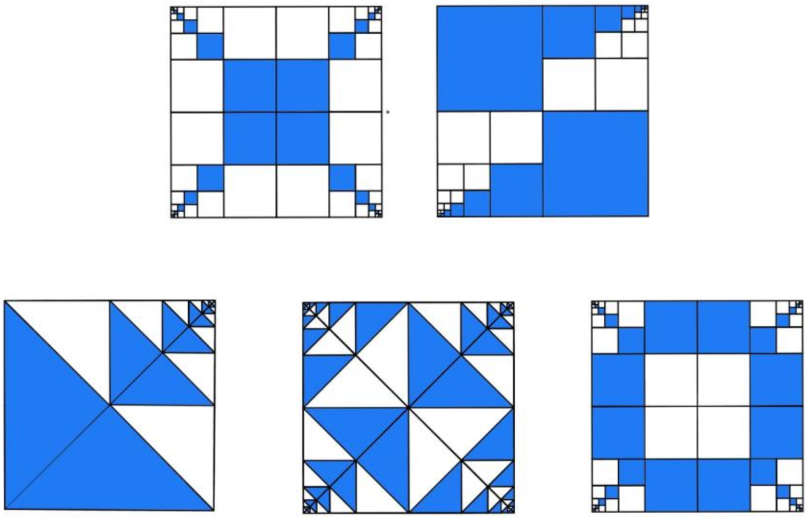


- In arts, many modern art-works can be created by using G.P.

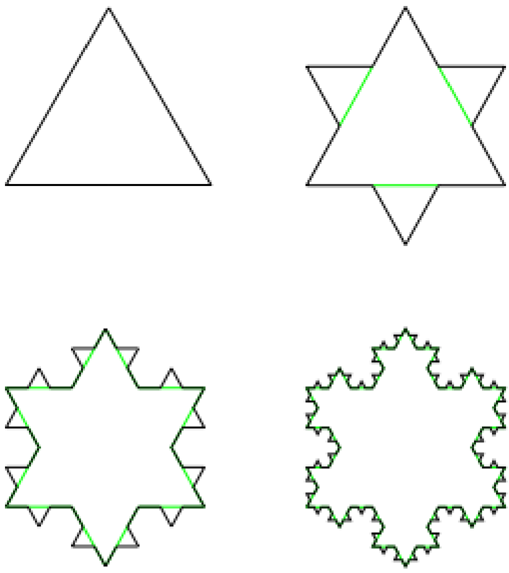
## Geometric Progressions

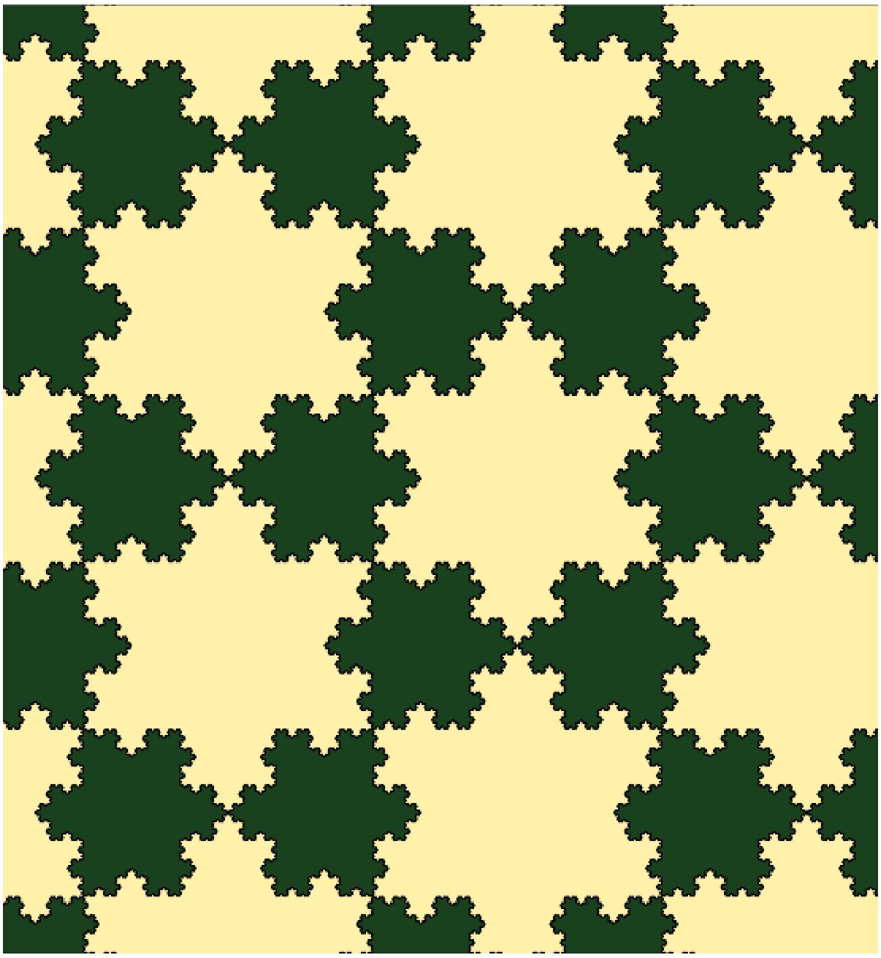
Each square is filled in with a pattern of squares or triangles that become smaller and smaller until they are infinitely small.

For each square below, work out what fraction of the square is shaded blue.



- Koch's Snowflakes have been made by iteration of geometrical figures which follow G.P. Fractal geometry is also based on G.P.





**Real World Problems**

**Example 11**: A person sent an e-mail to three of his friends and asks each one of them to forward the e-mail to three more friends with the same instructions to continue the chain in the same manner. Find the total number of e-mails at the end of $7^{th}$ set of mails.

**Solution**: Here number of mails are 3, 9, 27,…

which is a G.P. having a = 3, r =3 and n = 7

Total number of e-mails = $S_7 = \frac{a(r^6-1)}{r-1}$

$$\frac{3(3^6-1)}{3-1}$$

$$\frac{3(729-1)}{2}$$

$$\frac{3\times 728}{2}$$

$$3\times 364$$

$$1092$$

**Example 12**: A certain type of bacteria gets double in 5 hours. By using sum of series, calculate the total number of bacteria present at the end of 25 hours if starting bacteria were 100.

**Solution**: Here sequence is 100, 200, 400,…

which is a G.P. with a = 100, r = 2

Total number of bacteria = $S_5 = \frac{100(2^4-1)}{2-1} = 100 \times 15 = 1500$

## 9. Summary

- A sequence $a_{1,}a_2, a_3, \ldots.., a_n, \ldots.$ is called a geometric progression if each term is non-zero and $\frac{a_{k+1}}{a_k} = r(constant) for k \geq 1$
- For a G.P., 'a' is the first term and 'r' is the common ratio.
- We can also write a G.P. as $a, ar, ar^2, ar^3, ar^4, \ldots..$ where '$a$' is first term and $r \neq 0$ is the common ratio of G.P.
- General term of G.P. can be generalized as $a_n = a.r^{n-1}$
- Sum to n terms of geometric series is given by $S_n = \frac{(r^{n+1}-1)}{(r-1)}, r \neq 1$
- $S = S_\infty = \frac{a}{1-r}, |r| < 1$