1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 02 (Class XI, Semester - 2)
Module Name/Title	Binomial Theorem- Introduction, Pascal's Triangle, Proof of Binomial Theorem - Part 1
Module Id	kemh_20801
Pre-requisites	Knowledge of the Polynomials, Multiplication of Polynomials, Algebraic Identities, Factorial, Combinations, Principle of Mathematical Induction
Objectives	After going through this lesson, the learners will be able to understand the following: 1. What Pascal's Triangle is 2. Patterns in Pascal Triangle 3. Statement and Proof of Binomial Theorem 4. General term of a binomial expansion 5. Particular term of the expansion 6. Middle term(s) of an expansion 7. Coefficient of a given expression 8. Term independent of the variable
Keywords	Pascal's Triangle, Binomial Theorem, Factorial, Combinations, General Term, Middle Term, Coefficients,

2. Development Team

Role	Name	Affiliation
National MOOC Coordinator (NMC)	Prof. Amarendra P. Behera	CIET, NCERT, New Delhi
Program Coordinator	Dr. Rejaul Karim Barbhuiya	CIET, NCERT, New Delhi
Course Coordinator (CC) / PI	Dr. Til Prasad Sarma	DESM, NCERT, New Delhi
Course Co-Coordinator / Co-PI	Anjali Khurana	CIET, NCERT, New Delhi
Subject Matter Expert (SME)	Ms. Sangeeta Gulati	Department of Mathematics, Sanskriti School, New Delhi
Review Team	Prof. G.P. Dixit	Dr. B.R. Ambedkar University, Lucknow

Table of Contents:

1. Introduction

A **binomial** is a mathematical expression with two terms. For example, $2x-y,\ x^2-3, a+4b^3$ are all binomial expressions.

If we want to raise a binomial expression to a power higher than 2 (For example to find $(x-y)^{11}$) it is very cumbersome to multiply $x-y$ by itself 11 times. In this unit you will learn how to do such expansions much more efficiently. Firstly we will look at a triangular pattern of numbers, known as Pascal's Triangle, which can be used to obtain the expansions very quickly and then discuss the Binomial Theorem, which will give us a formula for expanding Binomial expressions. We will use the general term of the expansion to find a term given its' position. The position will be known directly or indirectly.

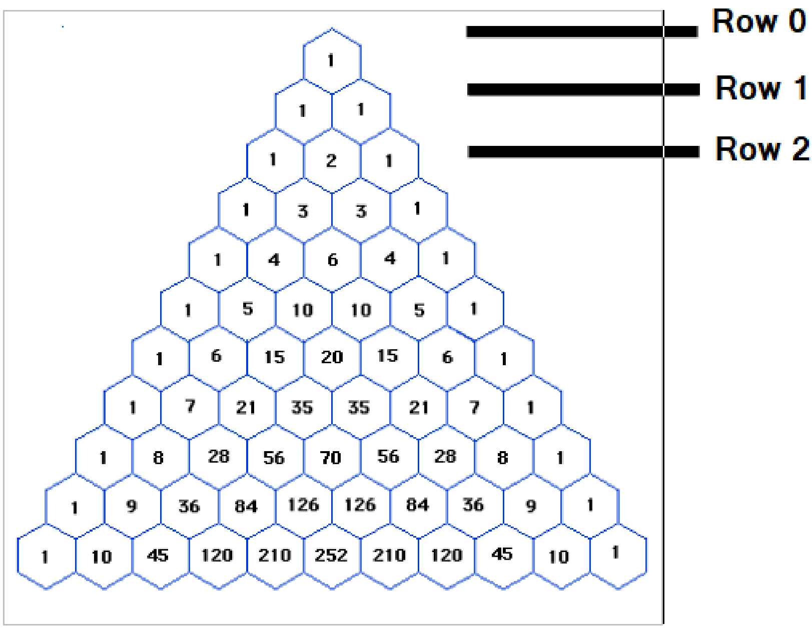
The Binomial Theorem states, that,

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_1a^{n-2}b^2 + \cdots + {}^nC_ra^{n-r}b^r + \cdots {}^nC_nb^n \text{ where , } Cr = \frac{n!}{(n-r)!r!}$$

Using this result, we will now consider the expansion of $(1+x)^n$

2. Pascal's Triangle

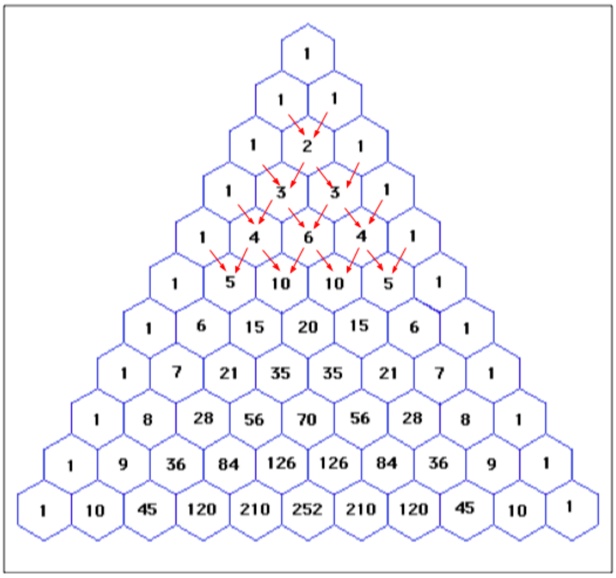
Let's study the following arrangement of numbers, what patterns do we see here?



(Note: Number of rows are taken from zero as further in this module it will be correlated with the index number)

Can we write the next row?

Yes, numbers do follow a pattern here; each row starts and ends with 1 also second element '2' in the row 2 is sum of the '1' and '1' f the row 1, elements of the row 4 are **1,4** (sum of 1st and 2nd element of the preceding row), **6** (sum of 2nd & 3rd element of the preceding row), **4** (sum of 3rd & 4th element of the preceding row) and **1**, so on.



So the next row in the above pattern is 1, 11, 55,165,330,462,462,330,165,55,11,1

This arrangement is called Pascal’s Triangle named after the French mathematician Blaise Pascal (1623-1662)It is also known as *Meru Prastara* by Pingla.

Pascal’s Triangle has some very interesting patterns, let’s note some of them.

- The rows read the same from left to right as right to left
- Sum of numbers in the rows are powers of 2
- Diagonal numbers are natural numbers and triangular numbers

What other patterns can you find here?Refer to the web links for this module to explore more patterns in Pascal’s Triangle.

Let’s consider following identities:

$(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Can you say what $(a+b)^4$ expansion is?

You may write,

$(a+b)^4 = (a+b)^2 (a+b)^2$

OR

$(a+b)^4 = (a+b)^3 (a+b)$

and use above identities to get the expansion.

Well, we are going to learn a more efficient method in this module based on the result known as Binomial Theorem.Binomial theorem is a useful ‘formula’ for finding any power of a binomial without multiplying at length. Before we state this theorem lets’ list only the coefficients of each expansion:

Index(Row)	Coefficients			
0	1			
1	1	1		
2	1	2	1	
3	1	3	3	1

These coefficients are following the pattern of Pascal's Triangle

So, when the index of the binomial is 4, the coefficients are 1 , 4 , 6, 4 and 1. To write the complete expansion, observe:

- The total number of terms in the expansion is one more than the index.
- Power of the first quantity 'a' is decreasing by 1, where as the power of the second quantity 'b' is increasing by 1 in each successive term
- In each term of the expansion, the sum of the indices of the variables a and b is same as the index of a + b.

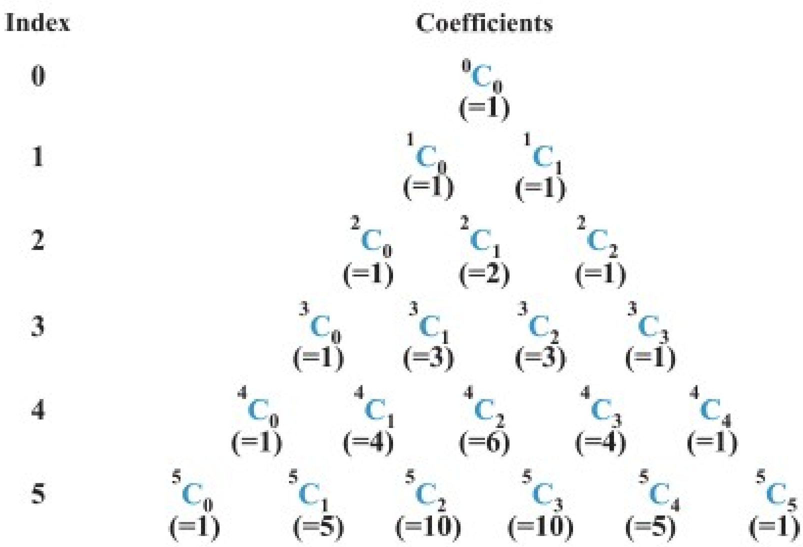
So, we have, $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

If we try to use above observations to write the expansion of $\left(2x-y^2\right)^{10}$, we will need to write the first 11 rows of Pascal's Triangle. This process can be time consuming and difficult if the index is a much bigger number. So, let's make some more connections with the numbers in the Pascal's Triangle with their position.

Recall, that $^{n}C_r = \dfrac{n!}{(n-r)!r!}, 0 \le r \le n$ and n is a non-negative integer, represents the number of ways r items can be selected from n given items.

Also, $^{n}C_0 = {}^{n}C_n = 1,$ for all n.

The Pascal's Triangle can now be written as :



(Image sourced from: NCERT Mathematics Text book, Class XI)

Following the above pattern, we can write the row of Pascal's Triangle for any index.For the index 10, we have(using Row 10 of Pascal's Triangle)

$^{10}C_0$ (=1) $^{10}C_1$ (=10) $^{10}C_2$ (=45) $^{10}C_3$ (=120) $^{10}C_4$ (=210) $^{10}C_5$ (=252)
$^{10}C_6$ (=210) $^{10}C_7$ (=120) $^{10}C_8$ (=45) $^{10}C_9$ (=10) $^{10}C_{10}$ (=1)

So,

$$\left(2x-y^2\right)^{10}=(2x)^{10}+10(2x)^9\left(-y^2\right)+45(2x)^8\left(-y^2\right)^2+120(2x)^7\left(-y^2\right)^3+210(2x)^6\left(-y^2\right)^4+252(2x)^5\left(-y^2\right)^5+$$
$$210(2x)^4\left(-y^2\right)^6+120(2x)^3\left(-y^2\right)^7+45(2x)^2\left(-y^2\right)^8+10(2x)\left(-y^2\right)^9+\left(-y^2\right)^{10}$$

Two very significant results can be noted from the Pascal's Triangle:

Result 1. $^nC_0={}^nC_n=1,\ \text{for all } n\in N$

Result 2. $^nC_{r-1}+{}^nC_r={}^{n+1}C_r, n\in N, 0\le r\le n$

Proof of Result2:

$$^n\mathrm{C}_r+{}^n\mathrm{C}_{r+1}=\frac{n!}{(n-r)!r!}+\frac{n!}{(n-r-1)!(r+1)!}$$

$$=\frac{n!}{(n-r)(n-r-1)!r!}+\frac{n!}{(n-r-1)!(r+1)(r!)}$$
$$=\frac{n!}{(n-r-1)!r!}\left[\frac{1}{n-r}+\frac{1}{r+1}\right]$$
$$=\frac{n!}{(n-r-1)!r!}\left[\frac{r+1+n-r}{(n-r)(r+1)}\right]$$
$$=\frac{n!}{(n-r-1)!r!}\left[\frac{n+1}{(n-r)(r+1)}\right]$$

$$^n\mathrm{C}_r+{}^n\mathrm{C}_{r+1}=\frac{n!}{(n-r)!r!}+\frac{n!}{(n-r-1)!(r+1)!}$$

$$=\frac{n!}{(n-r)(n-r-1)!r!}+\frac{n!}{(n-r-1)!(r+1)(r!)}$$
$$=\frac{n!}{(n-r-1)!r!}\left[\frac{1}{n-r}+\frac{1}{r+1}\right]$$

$$=\frac{(n+1)!}{(n-r)!(r+1)!}$$

$$\therefore {}^n\mathrm{C}_r+{}^n\mathrm{C}_{r+1}={}^{n+1}\mathrm{C}_{r+1}$$

Let us now generalize our observation and state and prove Binomial Theorem for positive integral index.

3. **Binomial Theorem**

Binomial Theorem (for positive integral index):

For any positive integer n, we have

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \cdots + {}^nC_ra^{n-r}b^r + \cdots + {}^nC_nb^n$$

where ${}^nC_r = \dfrac{n!}{(n-r)!r!}$

Proof:

We will prove Binomial Theorem using Principle of Mathematical Induction

$$Let\ \text{P(n)}: (a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \cdots + {}^nC_ra^{n-r}b^r + \cdots + {}^nC_nb^n$$

For n =1 ,

LHS of P(1)= $(a+b)^1 = a+b$

RHS of P(1)=${}^1C_0a^1 + {}^1C_1b^1 = a+b$

$\therefore P(1)$ is true.

Let P(k) be true.

$$\Rightarrow\ (a+b)^k = {}^kC_0a^k + {}^kC_1a^{k-1}b + {}^kC_2a^{k-2}b^2 + \cdots + {}^kC_kb^k$$

To show P(k+1) is true.

$$\text{i.e., } (a+b)^{k+1} = {}^{k+1}C_0a^{k+1} + {}^{k+1}C_1a^kb + {}^{k+1}C_2a^{k-1}b^2 + \cdots + {}^{k+1}C_{k+1}b^{k+1}$$

$$(a+b)^{k+1} = (a+b)(a+b)^k$$

$$= \ (a+b)\left({}^kC_0a^k + {}^kC_1a^{k-1}b + {}^kC_2a^{k-2}b^2 + \cdots + {}^kC_kb^k\ \right)$$

$$=\left({}^kC_0a^{k+1} + {}^kC_0a^kb + {}^kC_1a^kb + {}^kC_1a^{k-1}b^2 + {}^kC_2a^{k-1}b^2 + {}^kC_2a^{k-2}b^3 + \cdots + {}^kC_kab^k + {}^kC_kb^{k+1}\right)$$

$$=\left({}^kC_0a^{k+1} + \left({}^kC_0 + {}^kC_1\right)a^kb + \left({}^kC_1 + {}^kC_2\right)a^{k-1}b^2 + \left({}^kC_2 + {}^kC_3\right)a^{k-2}b^3 + \cdots + {}^kC_kab^k + {}^kC_kb^{k+1}\right)$$

Using :

Result1: ${}^n\mathrm{C}_0 = {}^n\mathrm{C}_n = 1$

Result 2: ${}^n\mathrm{C}_r + {}^n\mathrm{C}_{r+1} = {}^{n+1}\mathrm{C}_{r+1}$

$$\left(a+b\right)^{k+1}=\left({}^{k+1}C_0a^{k+1}+{}^{k+1}C_1a^kb+{}^{k+1}C_2a^{k-1}b^2+{}^{k+1}C_3a^{k-2}b^3+\cdots+{}^{k+1}C_{k+1}b^{k+1}\right)$$

$\therefore$ P(k+1) is true when P(k) is assumed to be true.

Hence by the principle of mathematical induction P(n) is true for all n $\in$N.

$$\left(a+b\right)^n={}^nC_0a^n+{}^nC_1a^{n-1}b+{}^nC_2a^{n-2}b^2+\cdots+{}^nC_ra^{n-r}b^r+\cdots+{}^nC_nb^n$$

$$\text{where } {}^n\mathrm{C}_r=\frac{n!}{\left(n-r\right)!r!}$$

Note:

1. The coefficients ${}^nC_r$, are called Binomial Coefficients
2. The general term in the expansion $\left(a+b\right)^n$ is given by $\mathrm{T}_{r+1}={}^nC_ra^{n-r}b^r$
3. In particular $\left(1+x\right)^n={}^nC_0+{}^nC_1x+{}^nC_2x^2+\ldots+{}^nC_rx^r+\ldots+{}^nC_nx^n$

Example 1: Using Binomial Theorem, expand $\left(3x+2y\right)^4$

$$\left(3x+2y\right)^4={}^4C_0\left(3x\right)^4+{}^4C_1\left(3x\right)^3\left(2y\right)+{}^4C_2\left(3x\right)^2\left(2y\right)^2+{}^4C_3\left(3x\right)\left(2y\right)^3+{}^4C_4\left(2y\right)^4$$

Solution: $=81x^4+216x^3y+216x^2y^2+96xy^3+16y^4$

Example 2: Compute $\left(1.3\right)^4$

Solution: Using Binomial Theorem, we have

$$\begin{aligned}\left(96\right)^3&=\left(100-4\right)^3\\&=\left(100+\left(-4\right)\right)^3\\&={}^3C_0\left(100\right)^3\left(-4\right)^0+{}^3C_1\left(100\right)^2\left(-4\right)^1+{}^3C_2\left(100\right)^1\left(-4\right)^2+{}^3C_3\left(100\right)^0\left(-4\right)^3\\&=1000000+3\left(10000\right)\left(-4\right)+3\left(100\right)\left(16\right)-64\\&=884736\end{aligned}$$

Example 3: Find the value of $\left(\sqrt{3}+\sqrt{2}\right)^6+\left(\sqrt{3}-\sqrt{2}\right)^6$

Solution:

$$\left(\sqrt{3}+\sqrt{2}\right)^6+\left(\sqrt{3}-\sqrt{2}\right)^6=\left(\sqrt{3}+\sqrt{2}\right)^6+\left(\sqrt{3}+\left(-\sqrt{2}\right)\right)^6$$

Using Binomial Theorem,we have

$$\left(\sqrt{3}+\sqrt{2}\right)^6 = {}^6C_0\left(\sqrt{3}\right)^6\left(\sqrt{2}\right)^0+{}^6C_1\left(\sqrt{3}\right)^5\left(\sqrt{2}\right)^1+{}^6C_2\left(\sqrt{3}\right)^4\left(\sqrt{2}\right)^2+{}^6C_3\left(\sqrt{3}\right)^3\left(\sqrt{2}\right)^3+$$
$${}^6C_4\left(\sqrt{3}\right)^2\left(\sqrt{2}\right)^4+{}^6C_5\left(\sqrt{3}\right)^1\left(\sqrt{2}\right)^5+{}^6C_6\left(\sqrt{3}\right)^0\left(\sqrt{2}\right)^6$$

and

$$\left(\sqrt{3}+\left(-\sqrt{2}\right)\right)^6 = {}^6C_0\left(\sqrt{3}\right)^6\left(-\sqrt{2}\right)^0+{}^6C_1\left(\sqrt{3}\right)^5\left(-\sqrt{2}\right)^1+{}^6C_2\left(\sqrt{3}\right)^4\left(-\sqrt{2}\right)^2+{}^6C_3\left(\sqrt{3}\right)^3\left(-\sqrt{2}\right)^3+$$
$${}^6C_4\left(\sqrt{3}\right)^2\left(-\sqrt{2}\right)^4+{}^6C_5\left(\sqrt{3}\right)^1\left(-\sqrt{2}\right)^5+{}^6C_6\left(\sqrt{3}\right)^0\left(-\sqrt{2}\right)^6$$

Note that the in the expansions, 2^{nd}, 4^{th} and 6^{th} terms are opposite in sign and when added they get cancelled. Thus,

$$\left(\sqrt{3}+\sqrt{2}\right)^6+\left(\sqrt{3}-\sqrt{2}\right)^6=\left(\sqrt{3}+\sqrt{2}\right)^6+\left(\sqrt{3}+\left(-\sqrt{2}\right)\right)^6$$
$$=2\left[{}^6C_0\left(\sqrt{3}\right)^6\left(\sqrt{2}\right)^0+{}^6C_2\left(\sqrt{3}\right)^4\left(\sqrt{2}\right)^2+{}^6C_4\left(\sqrt{3}\right)^2\left(\sqrt{2}\right)^4+{}^6C_6\left(\sqrt{3}\right)^0\left(\sqrt{2}\right)^6\right]$$
$$=2\left[27+270+8\right]$$
$$=610$$

4. General and Middle Term(s)

Recall,

Binomial Theorem (for positive integral index):

For any positive integer n, we have

$$(a+b)^n={}^nC_0a^n+{}^nC_1a^{n-1}b+{}^nC_2a^{n-2}b^2+\cdots+{}^nC_ra^{n-r}b^r+\cdots+{}^nC_nb^n$$

where ${}^nC_r=\dfrac{n!}{(n-r)!r!}$

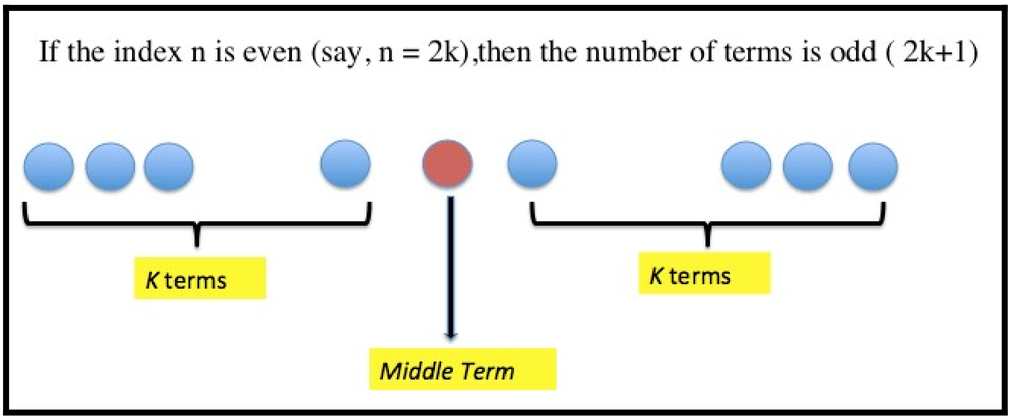
The general term in the expansion $(a+b)^n$ is given by $T_{r+1}={}^nC_ra^{n-r}b^r$

Note that the Binomial coefficient of (r+1)th term is given by ${}^nC_r$ and the sum of the powers of *a* and *b* is same as the index ‘n’ of the expansion.

Middle Term(s)

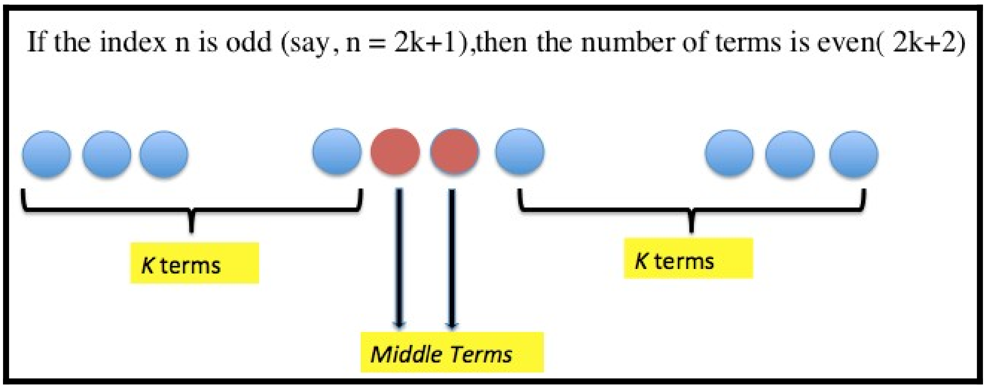
In the expansion $(a+b)^n$, we have

i. If n is even, then the number of terms in the expansion will be n + 1. Since n is even so n+1 is odd. Therefore the middle term is $\left(\frac{n+1+1}{2}\right)^{th}$ $i.e.\left(\frac{n}{2}+1\right)^{th}$ term



For example, in the expansion of $(3x-y^2)^{10}$, the middle term is $\left(\frac{10}{2}+1\right)^{th}$ $i.e., 6^{th}$ term.

ii. If n is odd, then the number of terms in the expansion will be n + 1. Since n is odd so n+1 is even. Therefore there will be two middle term in the expansion, namely, $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+1}{2}+1\right)^{th}$ term



For example, in the expansion of

$(2x^3+5y)^7$, the middle terms are $\left(\frac{7+1}{2}\right)^{th}$ and $\left(\frac{7+1}{2}+1\right)^{th}$ $i.e., 4^{th}$ term and 5^{th}term

Let’s try to use above observations:

Example 4: Find the 4^{th} term in the expansion of $\left(x^2+\frac{3}{y}\right)^7$

Solution:

We know that $T_{r+1} = {}^nC_r a^{n-r} b^r$

Here we have to find the 4th term,

So, $n = 7, r = 3, a = x^2, b = \frac{3}{y}$

$$\therefore\ T_4 = {}^7C_3 \left(x^2\right)^4 \left(\frac{3}{y}\right)^3$$
$$= \frac{7!}{3!4!}\left(x^8\right)\left(\frac{27}{y^3}\right)$$
$$= 945x^8y^{-3}$$

Example 5: Find the 3rd term from the end in the expansion of $\left(\frac{x}{3} - \frac{3}{x}\right)^7$

Solution:

There are 8 terms in the expansion of $\left(\frac{x}{3} - \frac{3}{x}\right)^7$**,** 3rd term from the end is (8 – 3 + 1)th term from the beginning.

Here we have to find the 6th term,

So, $n = 7, r = 5, a = \frac{x}{3}, b = -\frac{3}{x}$

$$T_6 = {}^7C_5\left(\frac{x}{3}\right)^2\left(-\frac{3}{x}\right)^5 = \frac{-567}{x^3}$$

Example 6: Find the middle term(s) in the expansion of :

i. $\left(2x^2 - 3y\right)^5$

ii. $\left(\frac{x}{3} - 3y^2\right)^8$

Solution:

i. There are 6 terms in the expansion of $\left(2x^2 - 3y\right)^5$

So, the middle terms are $\frac{6}{2} = 3^{rd}$ and 4^{th} terms.

$$T_{r+1} = {}^5C_r\left(2x^2\right)^{5-r}\left(-3y\right)^r$$

Thus, we have the middle terms , $T_3 = T_{2+1} = {}^5C_2\left(2x^2\right)^{5-2}\left(-3y\right)^2 \;= 720x^6y^2$

and, $T_4 = T_{3+1} = {}^5C_3\left(2x^2\right)^{5-3}\left(-3y\right)^3 \;= 720x^6y^2 = -1080x^4y^3$

ii. There are 9 terms in the expansion of $\left(\frac{x}{3} - 3y^2\right)^8$

So, the middle term is $\frac{9+1}{2} = 5^{th}$ term

$$T_{r+1} = {}^8C_r\left(\frac{x}{3}\right)^{8-r}\left(-3y^2\right)^r$$

Thus, we have middle term, $T_5 = T_{4+1} = {}^8C_4\left(\frac{x}{3}\right)^{8-4}\left(-3y^2\right)^4 = 70x^4y^8$

Example 7: Show that the middle term in the expansion of $\left(1+x\right)^{2n}$ is $\frac{1.3.5...(2n-1)}{n!}2^n x^n$

Solution:

The index of the expansion 2n is even and the numbers of terms in expansion are 2n+1 (odd).

The middle term of the expansion $\left(1+x\right)^{2n}$ is $\left(\frac{2n+1+1}{2}\right)^{th}$ $i.e.,\left(n+1\right)^{th}$ term.

$$T_{n+1} = {}^{2n}C_n\left(1\right)^n x^n = {}^{2n}C_n x^n$$

To get the required expression, we will simplify ${}^{2n}C_n$

$$\begin{aligned}{}^{2n}C_n &= \frac{(2n)!}{n!n!} = \frac{(2n)(2n-1)(2n-2)(2n-3)...4.3.2.1}{n!n!}\\ &= \frac{\left[(2n)(2n-2)(2n-4)...4.2\right].\left[(2n-1)(2n-3)...5.3.1\right]}{n!n!}\end{aligned}$$

(Note that in 2n! expansion, there are n even factors and n odd factors and 2 is a common factor from the n even factors)

$$\begin{aligned}{}^{2n}C_n &= \frac{2^n\left[(n)(n-1)(n-2)...2.1\right].\left[(2n-1)(2n-3)...5.3.1\right]}{n!n!}\\ &= \frac{2^n n!.\left[(2n-1)(2n-3)...5.3.1\right]}{n!n!}\\ &= \frac{2^n\left[(2n-1)(2n-3)...5.3.1\right]}{n!}\end{aligned}$$

Hence, the middle term of the expansion $(1+x)^{2n}$ is

$$T_{n+1} = {}^{2n}C_n x^n$$
$$= \frac{2^n\left[(2n-1)(2n-3)...5.3.1\right]}{n!}x^n$$

Example 8: Find a if the coefficients of x^2 and x^3 in the expansion of $(3+ax)^9$ are equal.

Solution:

General term in the expansion of $(3+ax)^9$ is $T_{r+1} = {}^9C_r 3^{9-r}(ax)^r = {}^9C_r 3^{9-r}(a)^r(x)^r$

Comparing indices of x in x^2 and in T_{r+1}, get $r = 2$

Coefficient of $x^2 = {}^9C_2(3)^7(a)^2$

Comparing indices of x in x^3 and in T_{r+1}, get $r = 3$

Coefficient of $x^3 = {}^9C_3(3)^6(a)^3$

Since the coefficients of x^2 and x^3 in the expansion of $(3+ax)^9$ are equal,

$${}^9C_2(3)^7(a)^2 = {}^9C_3(3)^6(a)^3$$
$$\Rightarrow a = \frac{9}{7}$$

Example 9: Find the coefficient of a^5b^7 in expansion of $(a-2b)^{12}$

Solution:

Let, a^5b^7 occurs at $(r+1)^{th}$ place in the expansion of $(a-2b)^{12}$

$$T_{r+1} = {}^{12}C_r a^{12-r}(-2b)^r = {}^{12}C_r(-2)^r a^{12-r}(b)^r$$

Comparing indices of a and b in a^5b^7 and in T_{r+1}, we get

$12-r = 5, i.e., r = 7$

$\therefore$ Coefficient of $a^5b^7 = {}^{12}C_7(-2)^7 = -101376$

Example 10: Find the term independent of x in the expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^6, x \neq 0$

Solution:

The term **independent of x** or the **absolute term** in the expansion $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^6$ is a term (if it exists) in which the index of x is 0.

Here,

$$T_{r+1} = {}^6C_r \left(\frac{3}{2}x^2\right)^{6-r} \left(\frac{-1}{3x}\right)^r$$

$$= {}^6C_r \left(\frac{3}{2}\right)^{6-r} \left(x^2\right)^{6-r} \left(\frac{-1}{3}\right)^r \left(\frac{1}{x}\right)^r$$

Collecting index of *x* and constant terms, we get

$$T_{r+1} = (-1)^r \, {}^6C_r \frac{(3)^{6-2r}}{(2)^{6-r}} (x)^{12-3r}$$

The term will be independent of *x* if the index of *x* is zero, i.e., $12 - 3r = 0$. Thus $r = 4$

Hence, the term independent of x in the expansion of

$\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^6$ is the 5^{th} term, given by $(-1)^4 \frac{(3)^{6-8}}{(2)^{6-4}} = \frac{5}{12}$

Example 11: Find the term independent of *x* (if it exists) in the expansion of

$\left(x^2 + \frac{3}{x}\right)^4, x \neq 0$

Solution:

The general term of the given expansion is,

$$T_{r+1} = {}^4C_r \left(x^2\right)^{4-r} \left(\frac{3}{x}\right)^r$$

$$= {}^4C_r (x)^{8-3r} (3)^r$$

The term will be independent of *x* if the index of *x* is zero, i.e., $8 - 3r = 0$ which gives $r = 8/3$

This is not possible as *r* is a natural number, hence the given expansion does not have any term independent of *x*.

5. Expansion of $(1+x)^n$

We have,

$$(1+x)^n = C_0 1^n + C_1 1^{n-1} x + C_2 1^{n-2} x^2 + \cdots + C_r 1^{n-r} x^r + \cdots C_n x^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_r x^r + \cdots C_n x^n$$

Here the general term, $T_{r+1} = {}^nC_r x^r$ and note that in each term of the expansion the coefficients are the same as the binomial coefficients. That is, the coefficient of x^r is C_r . In other words, coefficient of (r+1)th term is C_r

Let’s take some application of the above discussion:

Example 12: Find the value of r if the coefficient of (2r+5)th and (r-2)th terms in the expansion of $(1+x)^{16}$ are equal.

Solution:

Coefficient of $T_{2r+5} = Coefficient of T_{r-2}$

Coefficient of $T_{2r+5} = C_{2r+4}$

Coefficient of $T_{r-2} = C_{r-3}$

$\Rightarrow C_{2r+4} = C_{r-3} \Rightarrow either\, 2r+4 = r-3, i.e., r=-7, which is not possible as r > 0 \vee 2r+4+r-3=16, i.e., r=5$

Example 13: The coefficients of three consecutive terms in the expansion of $(1+x)^n$ are in the ratio 1:7:42. Find n.

Solution:

Let the three consecutive terms be $T_{r+1}, T_{r+2} T_{r+3}$

Then $C_r : C_{r+1} : C_{r+2} = 1:7:42$

$\Rightarrow C_r : C_{r+1} = 1:7 i.e., \frac{C_r}{C_{r+1}} = \frac{1}{7} \wedge C_{r+1} : C_{r+2} = 7:42 i.e., \frac{C_{r+1}}{C_{r+2}} = \frac{7}{42}$

$\frac{C_r}{C_{r+1}} = \frac{1}{7} \Rightarrow \frac{n!}{(n-r)!r!} \times \frac{(n-r-1)!(r+1)!}{n!} = \frac{1}{7} \frac{(r+1)}{n-r} = \frac{1}{7} \Rightarrow 7r+7 = n-r \therefore n-8r = 7 \cdots (i)$

$\frac{C_{r+1}}{C_{r+2}} = \frac{7}{42} \Rightarrow \frac{n!}{(n-r-1)!(r+1)!} \times \frac{(n-r-2)!(r+2)!}{n!} = \frac{1}{6} \frac{r+2}{n-r-1} = \frac{1}{6} \Rightarrow 6r+12 = n-r-1 \therefore n-7r = 13 \cdots (ii)$

Solving *(i)* and (*ii*) simultaneously we get r = 6 and n = 55.

Example 14: Show that the coefficient of the middle term in the expansion of $(1+x)^{2n}$ is equal to the sum of the coefficients of two middle terms in the expansion of $(1+x)^{2n-1}$

Solution:

Total number of terms in the expansion of $(1+x)^{2n}$ is 2n+1, which is an odd number. Thus, the middle term is the (n+1)th term, given by , ${}^{2n}C_n x^n$

Therefore, coefficient of the middle term in the expansion of $(1+x)^{2n}$ is

Now, in the expansion of $(1+x)^{2n-1}$ the total number of terms are 2n, an even number. Thus, the middle terms are nth and (n+1)th terms.

Coefficient of nth term is ${}^{2n-1}C_{n-1}$ and of (n+1)th term is ${}^{2n-1}C_n$

Therefore, the coefficient of the middle term in the expansion of $(1+x)^{2n}$ is equal to the sum of the coefficients of two middle terms in the expansion of $(1+x)^{2n-1}$

Example 15: If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1+x)^n$ are in A.P. Then find the value of n.

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_rx^r + \cdots {}^nC_nx^n$$

Solution:

Here, coefficient of 2nd = ${}^nC_r$

Coefficient of 3rd = ${}^nC_2$

And coefficient of 4th = ${}^nC_3$

Given that, ${}^nC_r$,${}^nC_2$ and ${}^nC_3$ are in A.P.

Therefore, 2 ${}^nC_2$ = ${}^nC_1$ + ${}^nC_3$

Example 16: Find the coefficient of x^4

Solution: We can write,

$$1+x+x^2+x^3=(1+x)+x^2(1+x)=(1+x)(1+x^2)$$

Coefficient of $x^4 = 1\times55+55\times11+330\times1=990$

6. Summary

- Pascal's Triangle is a triangular arrangement of numbers in which
  - The row numbers are 0,1,2,3,…
  - Number of elements in each row is one more than the number of the row
  - Each row starts and ends with 1 and other elements are sum of the two numbers above it.
  - Elements of each row correspond to the coefficients of the expansion of a binomial raised to the index same as the number of the row.
- Binomial Theorem for positive integral index n:

  $$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \cdots + {}^nC_ra^{n-r}b^r + \cdots + {}^nC_nb^n$$

  where ${}^nC_r = \dfrac{n!}{(n-r)!r!}$

- The coefficients nCr, are called Binomial Coefficients
- The general term in the expansion $(a+b)^n$ is given by $T_{r+1} = {}^nC_ra^{n-r}b^r$
- Binomial Theorem for positive integral index n:

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \cdots + {}^nC_ra^{n-r}b^r + \cdots + {}^nC_nb^n$$

$$\text{where } {}^nC_r = \frac{n!}{(n-r)!r!}$$

- The general term in the expansion $(a+b)^n$ is given by $T_{r+1} = {}^nC_ra^{n-r}b^r$
- Term independent of x is a term (if it exists) in which index of x is zero. This term is also called an absolute term.
- If n is even, then the middle term is $\left(\frac{n+1+1}{2}\right)^{th}$ *i.e.* $\left(\frac{n}{2}+1\right)^{th}$ term
- If n is odd, there will be two middle term in the expansion, namely, $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+1}{2}+1\right)^{th}$ term