1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 01 (Class XI, Semester - 1)
Module Name/Title	Permutation and Combination-Permutation - Part 3
Module Id	kemh_10703
Pre-requisites	Knowledge of the fundamental principle of counting. Differentiating between a problem with and without restrictions.
Objectives	After going through this lesson, the learners will be able to understand the following : ● Differentiate between permutation and combination problems ● Solve problems based on combinations using basic understanding ● Solve problems based on combinations by the use of formula.
Keywords	Fundamental Principle of counting, Product rule, Restrictions, Permutation and Combinations

2. Development Team

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1. Introduction

Till now we have studied about permutations and we are familiar with the concept that in permutations if there are n objects and there are r ways in which these n objects can be arranged, then the number of arrangements possible are given by $\frac{n!}{(n-r)!}$.

Let us take an example to revisit the concept :

- If there are 3 chairs and 5 people who will sit on those chairs then

CHAIR 1 can be occupied by any of the 5 people.

CHAIR 2 can be occupied by the remaining 4 people (as 1 has already sat)

CHAIR 3 can be occupied by any of 3 remaining people.

Thus the number of ways in which these 5 people can sit on the 3 chairs are $5 \times 4 \times 3 = 60\, ways$.

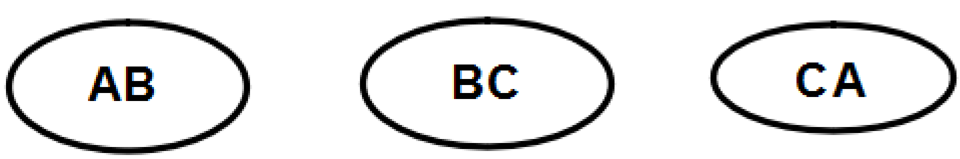
- In how many ways can the letters A, B, C be rearranged taking all the three at a time?

We can write it as ABC, ACB, BAC, BCA, CAB, CBA

2. Understanding Combinations

- Suppose there are 3 lawn Tennis players A, B, C. A team consisting of 2 players is to be formed. In how many ways can we do so? Is the team of A and B different from the team of B and A? Here, order is not important.

In fact, there are only 3 possible ways in which the team could be constructed.



- Suppose there are 12 children who have gone for a group project to a different work place. How many handshakes will take place in the room with these 12 students?

Twelve persons meet in a room and each shakes hand with all the others. How do we determine the number of hand shakes. X shaking hands with Y and Y with X will not be two different hand shakes. Here, order is not important. There will be as many hand shakes as there are combinations of 12 different things taken 2 at a time.

- If you go to a market and ask for a triple sundae ice-cream from the shopkeeper, he gives you an option of (1) Black current, Vanilla and strawberry OR (2) Mango, Raspberry and Vanilla. Here suppose you choose option 1.

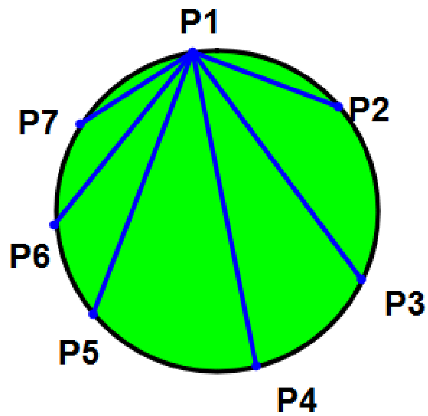
Now how he scoops the ice-cream in the bowl - Black current, Vanilla and strawberry

OR

Vanilla, Strawberry and Black current that does not matter. THIS IS COMBINATION as the order in which he scoops out the ice- cream does not matter.

 OR 

- Seven points lie on a circle. How many chords can be drawn by joining these points pairwise? There will be as many chords as there are combinations of 7 different things taken 2 at a time.



Chords from P1 to all the remaining points has been shown.

Suppose we have 4 different objects A, B, C and D. Taking 2 at a time, if we have to make combinations, these will be AB, AC, AD, BC, BD, CD. Here, AB and BA are the same combination as order does not alter the combination. This is why we have not included BA, CA, DA, CB, DB and DC in this list. There are as many as 6 combinations of 4 different objects taken 2 at a time, i.e., ${}^4C_2 = 6$.

Corresponding to each combination in the list, we can arrive at 2! permutations as 2 objects in each combination can be rearranged in 2! ways. Hence, the number of permutations = ${}^4C_2 \cdot 2!$

On the other hand, the number of permutations of 4 different things taken 2 at a time = ${}^4P_2$.

Therefore $\quad {}^4P_2 = {}^4C_2 \times 2! \quad \text{or} \quad \frac{4!}{(4-2)!\,2!} = {}^4C_2$

Now, let us suppose that we have 5 different objects A, B, C, D, E. Taking 3 at a time, if we have to make combinations, these will be ABC, ABD, ABE, BCD, BCE, CDE, ACE, ACD, ADE, BDE. Corresponding to each of these 5C3 combinations, there are 3! permutations, because, the three objects in each combination can be rearranged in 3 ! ways. Therefore, the total of permutations = $\mathbf{{}^5C_3 \cdot 3!}$

Therefore $\quad {}^5P_3 = {}^5C_3 \times 3! \quad \text{or} \quad \frac{5!}{(5-3)!\,3!} = {}^5C_3$

Thus the above result can be stated as

$$^{n}\mathrm{P}_{r} = {}^{n}\mathrm{C}_{r} \;\; r!, \; 0 < r \leq n.$$

Which means

Corresponding to each combination of $^{n}\mathrm{C}r$ we have $r!$ permutations, because r objects in every combination can be rearranged in r ! ways.

Hence, the total number of permutations of n different things taken r at a time is $^{n}\mathrm{C}_{r} \times r!$ Thus

$$^{n}\mathrm{P}_{r} = {}^{n}\mathrm{C}_{r} \times r!, \;\; 0 < r \leq n.$$

From above $\dfrac{n!}{(n-r)!} = {}^{n}\mathrm{C}_{r} \times r!$, i.e., $\quad {}^{n}\mathrm{C}_{r} = \dfrac{n!}{r!(n-r)!}$.

In particular, if $r = n$, $^{n}\mathrm{C}_{n} = \dfrac{n!}{n!\;0!} = 1$.

From the discussion we may conclude that :

1.

$$\frac{n!}{(n-r)!} = {}^{n}\mathrm{C}_{r} \times r!, \text{ i.e., } \quad {}^{n}\mathrm{C}_{r} = \frac{n!}{r!(n-r)!}$$

If $n = r$ then

$$\boxed{{}^{n}\mathrm{C}_{n} = \frac{n!}{n!\;0!} = 1.}$$

2. We define $^{n}\mathrm{C}_{0} = 1$, i.e., the number of combinations of n different things taken nothing at all is considered to be 1. Counting combinations is merely counting the number of ways in which some or all objects at a time are selected. Selecting nothing at all is the same as leaving behind all the objects and we know that there
is only one way of doing so. This way we define $^{n}\mathrm{C}_{0} = 1$.

3.

As $\dfrac{n!}{0!(n-0)!} = 1 = {}^{n}C_{0}$, the formula $^{n}\quad {}_{r} = \dfrac{n}{r\ (n-r)}$ is applicable for $r = 0$ also.

4. Also

$$^{n}C_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!} = {}^{n}C_{r}$$

selecting *r* objects out of *n* objects is same as rejecting (*n* – *r*) objects.

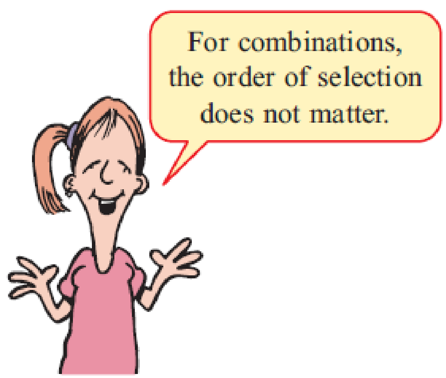
3. Differentiating a permutation from a combination

Order matters (PERMUTATION)   Order doesn't matter(COMBINATION)

Let us put it in this way:

PERMUTATION: A set of objects in which position is important. In a lock code 2 1 3 is different from 1 2 3.

COMBINATION: A set of objects in which position or order is not important.



The general formula is

$$C(n,k) = \frac{P(n,k)}{k!}$$

which means "Find all the ways to pick k people from n, and divide by the k! variants". Writing this out, we get our **combination formula**, or the number of ways to combine k items from a set of n:

$$C(n,k) = \frac{n!}{(n-k)!k!}$$

A few examples

Here’s a few examples of combinations (order doesn’t matter) and permutations (order matters).

- Combination: Picking a team of 3 people from a group of 10.

  C(10,3) = $\frac{10!}{3!(10-3)!}$ = 10 × 9 × 8 / (3 × 2 × 1) = 120.

Permutation: Picking a President, VP and Water-boy from a group of 10. P(10,3) = 10!/7! = 10 × 9 × 8 = 720.

- Combination: Choosing 3 desserts from a menu of 10. C(10,3) = 120.

  Permutation: Listing your 3 favorite desserts, in order, from a menu of 10. P(10,3) = 720.

Don’t memorize the formulas, understand why they work. **Combinations sound simpler than permutations, and they are. You have fewer combinations than permutations.**

Each of the different groups or selections formed by taking some or all of a number of objects is called a combination.

Sometimes, it will be clearly stated in the problem itself whether permutation or combination is to be used. However if it is not mentioned in the problem, we have to find out whether the question is related to permutation or combination.

Consider a situation where we need to find out the total number of possible samples of two objects which can be taken from three objects P, Q, R. To understand if the question is related to permutation or combination, we need to find out if the order is important or not.

If order is important, PQ will be different from QP, PR will be different from RP and QR will be different from RQ

If order is not important, PQ will be same as QP, PR will be same as RP and QR will be same as RQ

Hence,

If the order is important, problem will be related to permutations.

If the order is not important, problem will be related to combinations.

For permutations, the problems can be like "What is the number of permutations the can be made", "What is the number of arrangements that can be made", "What are the different number of ways in which something can be arranged", etc.

For combinations, the problems can be like "What is the number of combinations the can be made", "What is the number of selections theat can be made", "What are the different number of ways in which something can be selected", etc.

pq and **qp** are two different permutations, but they represent the same combination.

Mostly problems related to word formation, number formation etc will be related to permutations. Similarly most problems related to selection of persons, formation of geometrical figures, distribution of items (there are exceptions for this) etc will be related to combinations.

It is **very** important to make the distinction between permutations and combinations. In permutations, order matters and in combinations order does **not** matter. The important information can be summarized by:

	Order	Number
Permutation	matters	$P(n,k) = \frac{n!}{(n-k)!}$
Combination	does not matter	$C(n,k) = \frac{n!}{(n-k)!k!}$

Each of the different groups or selections formed by taking some or all of a number of objects is called a combination.

Examples

Suppose we want to select two out of three girls P, Q, R. Then, possible combinations are PQ, QR and RP. *(Note that PQ and QP represent the same selection.)*

Suppose we want to select three out of three girls P, Q, R. Then, only possible combination is PQR

4. Some Applications on Combinations

EXAMPLE 1

An educational committee of 4 has to be chosen from 7 men and 6 women. How many different committees can be chosen if

a) There are no restrictions.
b) There must be two men and two women.
c) There must be at least one man and a women.

SOLUTION : a) There are 7 + 6 = 13 people out of which we want 4 of them to form the committee.

Hence there are ^nC_r = $\frac{13!}{4!(13-4)!} = \frac{13!}{4!9!} = 715$ possible committees.

b) When we need 2 men from a group of 7 we have $^7C_2$ ways to choose from

2 women can be chosen from a group of 6 in $^6C_2$ ways

hence 2 men and 2 women can be chosen in

$^7C_2 \times {}^6C_2$ ways

$= \frac{7!}{2!5!} \times \frac{6!}{2!4!}$ = 315 possible committees.

C) To have a committee of at least 1 man and 1 women= We can have

= (1 man and 3 Women) or (2 men and 2 women) or (3 man + 1 women)

(Remembering the principle of Counting

$= ({}^7C_1 \times {}^6C_3) + ({}^7C_2 \times {}^6C_2) + ({}^7C_3 \times {}^6C_1)$

= 665 combinations

EXAMPLE 2:

Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?

SOLUTION:

Number of ways of selecting 3 consonants from 7 = $^7C_3$

Number of ways of selecting 2 vowels from 4 = $^4C_2$

Number of ways of selecting 3 consonants from 7 and 2 vowels from 4 = $^7C_3 \times {}^4C_2$

$$= \left(\frac{7 \times 6 \times 5}{3 \times 2 \times 1}\right) \times \left(\frac{4 \times 3}{2 \times 1}\right)$$

= 210

It means we can have 210 words where each word contains total 5 letters *(3 consonants and 2 vowels).*

Number of ways of arranging 5 letters among themselves

= 5!=5×4×3×2×1=120=5!=5×4×3×2×1=120

Hence, required number of ways

$= 210 \times 120 = 25200$

EXAMPLE 3: What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these

(i) four cards are of the same suit,

(ii) four cards belong to four different suits,

(iii) are face cards,

(iv) two are red cards and two are black cards,

(v) cards are of the same colour?

SOLUTION: There will be as many ways of choosing 4 cards from 52 cards as there are combinations of 52 different things, taken 4 at a time. Therefore

$$\text{The required number of ways} = {}^{52}C_4 = \frac{52!}{4!\ 48!} = \frac{49 \times 50 \times 51 \times 52}{2 \times 3 \times 4}$$

$$= 270725$$

(i) There are four suits: diamond, club, spade, heart and there are 13 cards of each suit. Therefore, there are ${}^{13}C_4$ ways of choosing 4 diamonds. Similarly, there are ${}^{13}C_4$ways of choosing 4 clubs, ${}^{13}C_4$ ways of choosing 4 spades and ${}^{13}C_4$ ways of choosing 4 hearts. Therefore

The required number of ways = ${}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4$

$$= 4 \times \frac{13!}{4!\ 9!} = 2860$$

(ii) There are13 cards in each suit.

Therefore, there are ${}^{13}C_1$ ways of choosing 1 card from 13 cards of diamond,

${}^{13}C_1$ ways of choosing 1 card from 13 cards of hearts, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of clubs, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of spades. Hence, by multiplication principle, the required number of ways are

$$= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13^4$$

(iii) There are 12 face cards and 4 are to be selected out of these 12 cards. This can be done in ${}^{12}C_4$ ways. Therefore, the required number of ways =

$$= \frac{12!}{4!\ 8!} = 495$$

(iv) There are 26 red cards and 26 black cards. Therefore, the required number of ways = ${}^{26}C_2$ × ${}^{26}C_2$.

$$= \left(\frac{26!}{2!\ 24!}\right)^2 = (325)^2 = 105625$$

(v) 4 red cards can be selected out of 26 red cards in ${}^{26}C_4$ ways.

4 black cards can be selected out of 26 black cards in ${}^{26}C_4$ ways.

Therefore, the required number of ways = ${}^{26}C_4$ + ${}^{26}C_4$

$$= 2 \times \frac{26!}{4!\ 22!} = 29900.$$

5. Summary

- Combination is an arrangement of particular objects where the order in which they appear does not matter.
- The combination reduces the number of redundancies hence there are fewer combinations than permutations.
- The formula of combination is given by ${}^{n}C_{r}$, The number of combinations of *n* different things taken *r* at a time, denoted by

$$ {}^{n}C_{r} = = \frac{n!}{r!(n-r)!},\ 0 \le r \le n.$$