1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 01 (Class XI, Semester - 1)
Module Name/Title	Permutation and Combination- Part 2
Module Id	kemh_10702
Pre-requisites	Knowledge of the fundamental principle of counting
Objectives	After going through this lesson, the learners will be able to understand the following : ● Understand the need for formulas in solving problems based on Permutation and Combination. ● Understand the Factorial notation. ● Apply the concept of factorial in solving questions based on permutations.
Keywords	Permutations, restrictions, grouping

2. Development Team

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1. Introduction

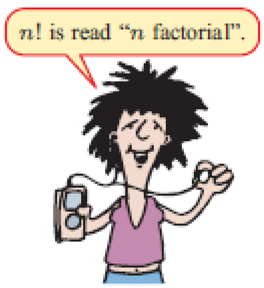
In our previous module we had seen that in problems involving counting principle, the product of consecutive natural number are quite common. **For example** $4\times 3\times 2\times 1$, $9\times 8\times 7\times 6$ thus for convenience the factorial notation has been introduced.

2. The Factorial Notation

In permutation and combination the factorial notation denoted by $n!$ holds significant importance. It is denoted by the product of first n natural numbers.

In other words if there are **n** objects and **n** places: then there are

$n \times (n-1) \times (n-2) \times (n-3) \times \ldots 4\times 3\times 2\times 1$ ways of arranging these object



It can be written in any order.

Notice that $8\times 7\times 6$ can also be written as

$$8\times 7\times 6 = \frac{8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{5\times 4\times 3\times 2\times 1} = \frac{8!}{5!}$$

An alternative recursive definition of factorial can also be stated as

$$n! = n\times (n-1)! \quad \text{for } n \geqslant 1$$

Let us see how

Since n! $= n \times (n-1) \times (n-2) \times (n-3) \times \ldots 4\times 3\times 2\times 1$

Also n! = n $\times$ [(n -1) $\times$ (n- 2) $\times$ (n -3) x … 4$\times$ 3$\times$ 2$\times$1]

= n $\times$ (n – 1)!

For example:

$4! = 4 \times 3 \times 2 \times 1 = ¿\ 4 \times 3$

3!= 3 $\times$ 2 $\times$1 = **3 $\times$ 2!**

2 ! = 2 $\times$1 = 2 $\times$1!

1 ! = 1$\times$0 ! (Since the value is 1 thus the value of 0 ! Is assumed to be 1)

Thus for completeness we define 0 ! = 1

3. Solving Factorials

If the factorials involve large numbers then cancelling the factors becomes important before evaluating the factorial:

$$\frac{300!}{297!} = \frac{300 \times 299 \times 298 \times 297!}{297!} = 300 \times 299 \times 298$$
$$= 26\,730\,600.$$

Example: Find 5 ! - 3 !

Solution:

<u>*Method 1*</u>: 5 $\times$ 4 $\times$ 3 $\times$ 2 $\times$ 1 - 3 $\times$ 2 $\times$ 1

= 120 – 6 = 114

<u>*Method 2:*</u> 5 $\times$ 4 $\times$ 3 ! - 3 !

= 3 $!(20-1)$ { Taking common}

= 6 $\times$ 19 = 114

Example: Find $\frac{7!}{5! \times 2!}$

Solution:

To solve this we $\frac{7!}{5! \times 2!} = \frac{7 \times 6 \times 5!}{5! \times 2 \times 1}$

$$\frac{7!}{5! \times 2!} = \frac{7 \times 6 \times \cancel{5!}}{\cancel{5!} \times 2 \times 1}$$

= 21

Example: Solve $\frac{1}{8!} - \frac{1}{9!} = \frac{x}{10!}$

To solve this, we will expand in such a manner that we get a common factorial in both LHS and RHS.

Hence we write $9! = 9 \times 8!$ and $10! = 10 \times 9 \times 8!$

We have $\frac{1}{8!} - \frac{1}{9!} = \frac{x}{10!}$

Or $\frac{1}{8!} - \frac{1}{9 \times 8!} = \frac{x}{10 \times 9 \times 8!}$

Or $\frac{1}{1} - \frac{1}{9} = \frac{x}{90}$ (cancelling 8! From both sides)

Or $\frac{9-1}{9} = \frac{x}{90}$

Or $\frac{8}{9} = \frac{x}{90}$

Thus, $x = 80$

4. Permutation

Permutation is an arrangement of objects in a specific order, where either all the objects or some of them can be considered at a time.

Example : In how many ways can the letters A, B, C be arranged if

a) 2 letters are taken at a time.

The letters can be arranged as AB, BA, BC, CB, AC, CA i.e. 6 ways

b) 3 letters are taken at a time

The letters can be arranged as ABC, ACB, BAC, BCA, CAB, CBA i.e. 6ways

Example : How many arrangement of the letters W,X,Y, Z are possible when 4 letters have to be taken each time?

The following arrangement of the letters will be possible:

WXYZ	WXZY	WYXZ	WYZX	WZXY	WZYX
XWYZ	XWZY	XYWZ	XYZW	XZYW	XZWY
YWXZ	YWZX	YXWZ	YXZW	YZWX	YZXW
ZWXY	ZWYX	ZXWY	ZXYW	ZYWX	ZYXW

Thus the following 24 arrangements are possible.

5. There are basically two types of permutation:

- **<u>Repetition is Allowed</u>**: For example: A lock can have a code by using any 3 digits and the digits can be repeated, let the code be 555, here we are allowed to repeat any digit from 0 to 9, three times.
- **<u>No Repetition</u>**: For example the first three people in a running race. One person cannot be first *and* second both.

5.1. Permutations with Repetition

These are the easiest to calculate.

When we have ***n*** things to choose from ... we have ***n*** choices each time!

When choosing ***r*** out of them, the permutations are:

n × n × ... *(r times)*

(In other words, there are **n** possibilities for the first choice, THEN there are **n** possibilites for the second choice, and so on, multiplying each time.)

Which is easier to write the expression down using an exponent of **r**:

n × n × ... (r times) = $\mathbf{n^r}$

Example: In a combination lock, there are 10 numbers to choose from (0,1,...9) and we choose any 3 of them:

10 × 10 × 10 = $\mathbf{10^3}$ **= 1,000 permutations**

So, the formula is simply:

$\mathbf{n^r}$
where ***n*** is the number of things to choose from, and we choose ***r*** of them (Repetition allowed, order matters)

<u>Permutations without Repetition</u>

In this case, we have to **reduce** the number of available choices each time.

For example, what order could 16 pool balls be in?

After choosing, say, number "14" we can't choose it again.

So, our first choice has 16 possibilites, and our next choice has 15 possibilities, then 14, 13, etc. And the total permutations are:

16 × 15 × 14 × 13 × ...3 $\times 2 \times 1$ **= 20,922,789,888,**000

But may be we don't want to choose them all, just 3 of them, so that is only:

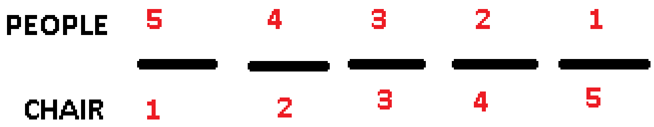
16 × 15 × 14 = 3,360

In other words, there are 3,360 different ways that 3 pool balls could be arranged out of 16 balls.

Without repetition our choices get reduced each time.

Example: There are 5 people and there are 5 chairs on which these people will be seated, how many ways can these people sit on the 5 chairs?

Let us number the chairs as 1, 2, 3, 4 , 5



and there be 5 people A ,B, C, D, E who would be seated on these chairs.

CHAIR 1: Any one out of the 5 people A,B,C,D,E can sit. Hence there are 5 options

CHAIR 2: Since one person has already sat on seat 1 (say A) we have 4 people remaining who can sit on chair 2.

CHAIR 3: Since 2 people have already sat on the chair 1, 2 thus we have 3 options left.

CHAIR 4: We have only two options of two people to sit.

CHAIR 5: We have only one option now.

$5 \times 4 \times 3 \times 2 \times 1 = 120$ ways in which people can be seated.

Thus if there are 5 people and 5 chairs there are

$5 \times 4 \times 3 \times 2 \times 1 = 120$ ways in which they can sit.

On generalizing we get

if there are n objects and all n are to be arranged

$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots 3 \times 2 \times 1$ ways of arrangement.

Let's now consider another situation if the number of chairs had been less.

Example: What if there are 3 chairs and 5 people who have to sit on those chairs.

CHAIR 1: Any one of the 5 people can sit.

CHAIR 2: Any one out of the 4 remaining people can sit.

CHAIR 3: Any one out of the three remaining people can sit

5 4 3 = 5 x 4 x 3 = 60

In the above example, we can write 5 $\times 4 \times 3 \, as$

$$5 \times 4 \times 3 = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \frac{5!}{2!} = \frac{5!}{(5-3)!}$$

In general, if we arrange n different objects taken r at a time, then the number of ways is

$$\mathbf{n \times (n-1) \times (n-2) \times \ldots (n-r+1)}$$

This can be written as

$$\frac{n \times (n-1) \times (n-2) \times \ldots (n-r+1) \times (n-r) \times (n-r-1) \ldots \times 3 \times 2 \times 1}{(n-r) \times (n-r-1) \ldots \times 3 \times 2 \times 1} = \frac{n!}{(n-r)!}$$

The number of **permutations** on n distinct symbols taken r at a time is:

$$\underbrace{n \times (n-1) \times (n-2) \times \ldots . \times (n-r+1)}_{r \text{ of these}} = \frac{n!}{(n-r)!}$$

This we also denote by the symbol ^nP_r

Hence, $^nP_r = \dfrac{n!}{(n-r)!}$ **where r > 0 and r** $\leq n$

Also , there may be a possibility that we have n objects and no way of arranging them, or we leave their arrangement unchanged. How many ways are possible then?

Simple answer there is only one such way.

Thus

$$^nP_0 = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$$

<u>Example:</u>

An advisory education committee comprises of 16 members, in how many ways can the top five positions be filled in the competition ladder.

<u>Solution:</u>

16	15	14	13	12
1st	2nd	3rd	4th	5th

Any one of the sixteen members can fill the first position.

Any one of the remaining 15 members can fill the second position.

Any one of the remaining 14 members can fill the third position.

Any one of the remaining 13 members can fill the fourth position.

Any one of the remaining 12 members can fill the fifth position.

In other words the first five position can be filled in:

$$^{16}P_5 = \frac{16!}{(16-5)!} = \frac{16!}{11!} = \mathbf{52410 \text{ ways.}}$$

6. Solving Problems when restrictions are involved.

We might come across problems when there are restrictions involved, in such cases we have to consider the restrictions first.

$$^{n}P_0 = \boxed{\frac{n!}{(n-0)!} = \frac{n!}{n!}} = 1$$

<u>**Example:**</u> **There are 6 different subject books that have to be kept together on the shelf. What are the different ways in which the books can be kept provided that books labeled with A and B have to be together.**

<u>**Solution:**</u>

Method 1 – Visualize – We see that we can have any of these 10 ways in which we can arrange AB or BA together.

$$\left.\begin{array}{cccccc}
A & B & \times & \times & \times & \times \\
B & A & \times & \times & \times & \times \\
\times & A & B & \times & \times & \times \\
\times & B & A & \times & \times & \times \\
\times & \times & A & B & \times & \times \\
\times & \times & B & A & \times & \times \\
\times & \times & \times & A & B & \times \\
\times & \times & \times & B & A & \times \\
\times & \times & \times & \times & A & B \\
\times & \times & \times & \times & B & A
\end{array}\right\} \text{These 10 ways are possible}$$

Once the arrangement of AB is decided as above (10 ways), the remaining 4 books can be arranged in 4! Ways

Thus there are $10 \times 4! = 10 \times 24 = 240$ ways of arranging the books.

METHOD 2 - A, B can be put together in (AB or BA) = 2! Ways.

Treating this pair as one. The remaining books (including the pair) can be arranged in 5! ways.

Thus the total arrangements are $2! \times 5! = 240$ ways

Strategy for objects together,
1) Treat the objects together as 1, determine the number of arrangements
2) For each group that is together, find the number of "internal" arrangements

2.2 Permutation when all objects are not distinct objects: Suppose we have a word where all the letters are not distinct: For example : BOOK. Here there are 2 O's. Had the letters been distinct as O_1 and O_2 there would have been 4! ways of arrangement ie that means O_1 and O_2 would have been treated as two different objects, let us see their arrangements when both the O are treated as same: Let us construct the table to visualize the total .

The number of permutations when all the objects are not distinct is given by

$\frac{4!}{2!}$ **in other words we observe that**

O_1KO_2B and O_2KO_1B is the same as OKOB

Arrangements when **O_1 and O_2 are treated as different**	**When O_1 and O_2 are treated as same**
B **O_1 O_2 K**	
B **O_2 O_1 K**	BOOK
K **O_1 O_2 B**	KOOB
K **O_2 O_1 B**	
O_2 O_1 B K	OOBK
O_1 O_2 B K	
O_2 O_1 K B	OOKB
O_1 O_2 KB	
K O_1 B O_2	KOBO
K O_2 B O_1	
B O_1 K O_2	BOKO
B O_2 K O_1	
B K O_1 O_2	BKOO
B K O_2 O_1	
K B O_1 O_2	KBOO
K B O_2 O_1	
O_1B K O_2	OBKO
O_2 B K O_1	
O_1 K B O_2	OKBO
O_2 K B O_1	
O_1 B O_2 K	OBOK
O_2 B O_1 K	

The number of permutations of n objects where p objects are of the same kind and the remaining are different is given by

$$\boxed{\frac{n!}{p!}}$$

Generalising the expression we can say that:

The number of permutations of n objects where p_1 objects are of same kind, p_2 objects are of same kind, p_3 objects are of same kind….p_k objects are of same kind, and the remaining are different then the number of possible arrangements are

$$\boxed{\frac{n!}{p_1!\,p_2!\,p_3!\ldots.p_k!}}$$

Example: In how many ways can the letters of the word ALLAHABAD be arranged?

__**Solution:**__ There are 9 letters where there are 4 A's, 2 L's and all the remaining letters are all different hence the number of arrangements will be

$\frac{9!}{4!2!}$ = **7560 ways.**

Example: The alphabets A, B ,C, D and E are placed in front of you.

a) How many different permutations could you have?

b) How many permutations end in C?

c) How many permutations can have the form

...	A	...	B	...	?

d) How many begin and end with a vowel?

Solution:

a) Since there are 5 letters the number of permutations possible are 5! = 120

b) **Since the last letter has to be C we have 1 way, the remaining 4 letters can be arranged in 4!** $\times 1$ **= 24 ways**

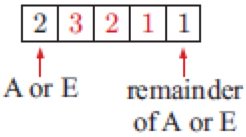
4	3	2	1	C
				(1 way)

c) **Since the 2nd and 4th place have to be filled with A and B respectively, (1 and 1 way each) the remaining places can be filled in 3** $\times 1 \times 2 \times 1 \times 1$ **= 6 ways**

3	1	2	1	1
	A		B	

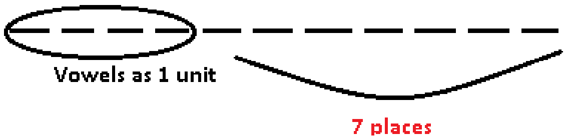
d) **Here there are 2 vowels A and E, Either of them can take the first position, the other one will go to the last place.**

The other places can be filled by 3! = 3 $\times 2 \times 1$ **ways, hence there are 2** $\times$ **3** $\times 2 \times 1$ **= 12 ways . (2 in the first place represents the 2 vowels)**



Example: In how many ways can the word MATHEMATICS be arranged such that the vowels are always together?

Solution: In the word ‘ MATHEMATICS’ there are 11 letters. There are 4 vowels A,E, A, I. These 4 vowels must come together so we treat them as 1 unit.



Hence in all the total number of letters are 8 (Vowels being taken as 1 unit and the 7 blanks, hence 1 + 7 = 8 letters) . Where there are 2 M’s, 2 T’s and all the other letters are different.

Hence we have $\frac{8!}{2!2!}$ = 10080 ways to arrange the letters

Also in the 4 vowels, we have two A’s and the vowels can also arrange themselves in

$\frac{4!}{2!}$ = 12 ways

Hence the total ways in which the arrangement are

10080 $\times$ 12 = 120960

Example: How many arrangements of the letters of the word ‘BENGALI’ can be made

(i) If the vowels are never together.

(ii) If the vowels are to occupy only odd places.

Solution:

There are 7 letters in the word ‘BENGALI’ of these 3 are vowels and 4 consonants.

(i) Considering vowels A, E, I as one letter, we can arrange the remaining letters in 5! ways in each of which vowels are together. These 3 vowels can be arranged among themselves in 3! ways.

Therefore, total number of words = 5! × 3!

= 120 × 6 = 720

So there are total of 720 ways in which vowels are ALWAYS TOGETHER.

Now,

Since there are no repeated letters, the total number of ways in which the letters of the word 'BENGALI' can be arranged:

$=7! = 5040=$

So,

Total no. of arrangements in which vowels are never together

= Number of all the arrangements possible – number of arrangements in which vowels are ALWAYS TOGETHER

$=5040-720 = 4320$

(ii) There are 4 odd places and 3 even places. 3 vowels can occupy 4 odd places in $^4P_3$ ways and 4 constants can be arranged in $^4P_4$ ways.

Hence , total Number of words $= {}^4P_3 \times {}^4P_4$ = **576**

7. Summary

Let us recap what we did in this unit:

- The factorial notation is denoted by n!and it represents the product of consecutive natural numbers from 1 to n i.e. $n! = n \times (n-1) \times (n-2)\ldots 4 \times 3 \times 2 \times 1$
- Permutations represent arrangement of objects in a definite order.
- $^nP_r = \dfrac{n!}{(n-r)!}$ **where r** ≥ 0 **and r** $\leq n$
- Permutations are with restrictions and without restrictions.
- There are permutations where there are various conditions applicable. Then the conditions need to be taken into consideration first.
- If there are permutations which involve pairing or grouping, then the pair or the group has to be treated as one unit and the others objects as distinct units.
- The number of permutations of n objects where p objects are of the same kind and the remaining are different is given by

  $$\boxed{\frac{n!}{p!}}$$

- The number of permutations of n objects where p_1 objects are of same kind, p_2 objects are of similar kind, p_3 objects are of similar kind….p_k objects are of similar kind and rest are different then the number of arrangements possible are

$$\boxed{\frac{n!}{p_1!\,p_2!\,p_3!\ldots.p_k!}}$$