## 1. Details of Module and its structure

| Module Detail | |
|---|---|
| Subject Name | Mathematics |
| Course Name | Mathematics 1 (Class XI, Semester - 1) |
| Module Name/Title | Complex Numbers – Argand Plane and Polar Representation; Modulus and Conjugate: Part 2 |
| Module Id | Kemh_10502 |
| Pre-requisites | • Representation of a point on the coordinate plane<br>• Pythagoras Theorem<br>• Basic trigonometric ratios in a right angle triangle |
| Objectives | After going through this lesson, the learners will be able to understand the following:<br>**1.** geometrical representation of a complex number<br>**2.** understand different properties of complex numbers and its representation on Argand plane<br>**3.** solve different mathematical problems using Argand plane<br>**4.** understand the polar or trigonometric form of a complex number<br>**5.** find the modulus of a complex number<br>**6.** find the argument of a complex number<br>**7.** identify the conjugate of a complex number and familiarized with its properties<br>**8.** identify the modulus of a complex number and familiarized with its properties |
| Keywords | Real Numbers; Imaginary; Complex Numbers; Argand Plane; Polar Coordinates; Conjugate; Argument; Modulus |

## 2. Development Team

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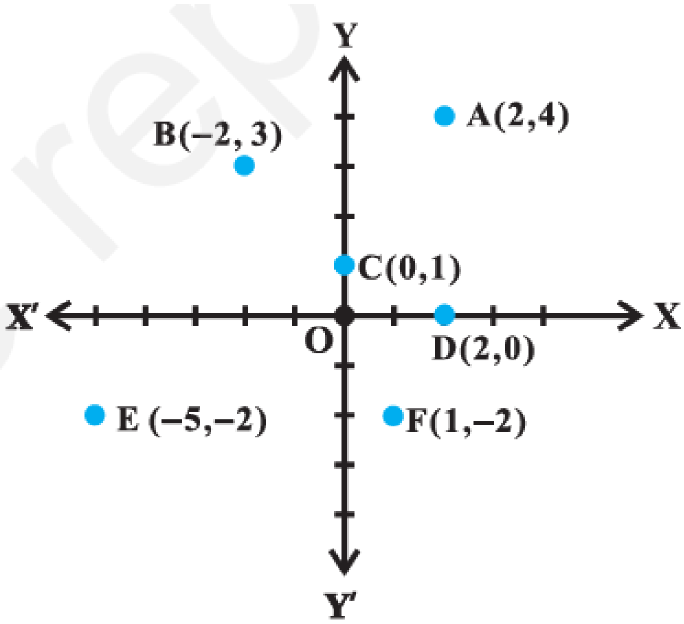
## 1. Introduction

In 1806, Argand published a way of representing Real Numbers and Imaginary Numbers by a diagram using two axes at right angles to each other like the Cartesian Plane. With the $x$ axis representing the Real Axis and the $y$ axis representing the Imaginary Axis, this is called the Argand Plane or the Complex Plane.

## 2. The Complex Plane or the Argand Plane

Some complex numbers such as $2 + 4i$, $-2 + 3i$, $0 + 1i$, $2 + 0i$, $-5 - 2i$ and$1 - 2i$ which correspond to the ordered pairs (2, 4), ( – 2, 3), (0, 1), (2, 0), ( –5, –2), and (1,–2) respectively, have been represented geometrically by the points A, B, C, D, E, and F, respectively in the adjoining figure.



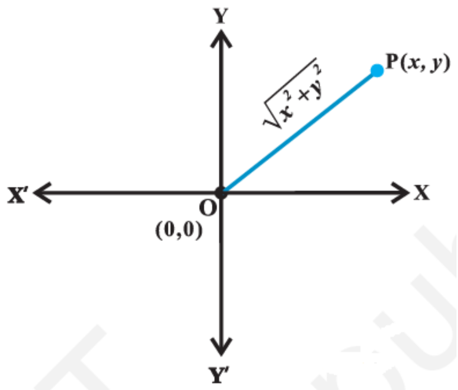

The plane having a complex number assigned to each of its point is called the *complex plane* or the *Argand plane*.

## 3. Modulus, Argument and Conjugate of a Complex Number

In the Argand plane, the modulus of the complex number is defined

as $|z| = |x + iy| = \sqrt{x^2 + y^2}$

which is the distance between the point P($x$, $y$) and the origin O (0, 0). The points on the $x$-axis correspond to the complex numbers of the form $a + i0$ and the points on the $y$-axis correspond to the complex numbers of the form $0 + ib$.



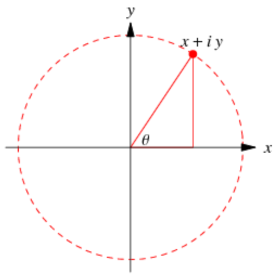

The angle $'\theta'$ from the positive axis to the line segment in the anticlockwise direction is called the **argument** of the complex number $z$.

Using trigonometry,

$$\tan\theta = \frac{y}{x}$$

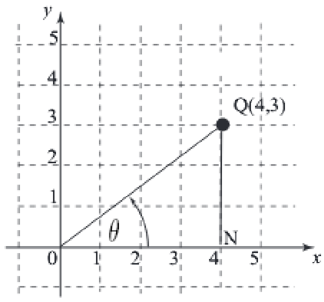
And hence,

$$\arg z = \theta = \tan^{-1}\frac{y}{x}$$




**Example:** Find the modulus and argument of $z = 4 + 3i$.

**Solution:** The complex number $z = 4 + 3i$ is shown in the figure. It has been represented by the point Q which has coordinates (4,3). The modulus of $z$ is the length of the line OQ




which we can find using Pythagoras' theorem.

$(OQ)^2 = 4^2 + 3^2 = 16 + 9 = 25$

and hence OQ= 5

$$\therefore \ |z| = 5$$

To find the argument we must calculate the angle between the positive direction of $x$-axis and the line segment OQ in the anti-clockwise direction. We have labelled this $\theta$ in the figure.

By referring to the right-angled triangle OQN in the figure, we see that

$$\tan\theta = \frac{3}{4}$$

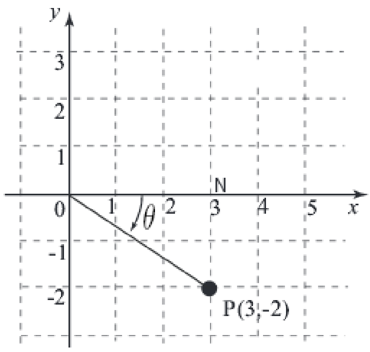
$$\arg z = \theta = \tan^{-1}(\frac{3}{4})$$

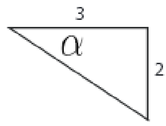
When the complex number lies in the first quadrant, calculation of the modulus and argument is straight forward. For complex numbers outside the first quadrant we need to be a little bit more careful. Consider the following example.

**Example**: Find the modulus and argument of

*z = 3 − 2i.*

**Solution**: The Argand diagram is shown in the figure. The point P with coordinates (3,−2) represents $z = 3 - 2i$.




We use Pythagoras' theorem in triangle ONP to find the modulus of *z*

$(OP)^2 = 3^2 + 2^2 = 13$

OP=√13

Using the symbol for modulus, we see that in this example

$$|z| = \sqrt{13}$$

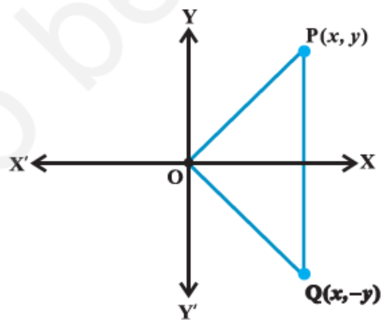
We must be more careful with the argument. When the angle θ shown in the figure is measured in a clockwise sense convention dictates that the angle is negative. We can find the size of the angle by referring to the right-angled triangle shown.

Here $\theta$ is the argument of $z = 3 - 2i$,

satisfying $tan\theta = -\frac{2}{3}$, where $cos\theta = \frac{3}{\sqrt{13}}$ $and\ sin\theta = -\frac{2}{\sqrt{13}}$

## <u>Conjugate of a Complex Number</u>

The representation of a complex number $z = x + iy$ and its **conjugate** $\bar{z} = x - iy$ in the Argand plane are, respectively, the points P $(x, y)$ and Q $(x, -y)$.Geometrically, the point $(x, -y)$ is the mirror image of the point $(x, y)$ on the real axis.



## 4. Polar form of a Complex Number

You will have already seen that a complex number is of the form $z = a + bi$. This form is called **Cartesian form**.

A point P in the complex plane can also be represented uniquely by using its polar coordinates $(r, \theta)$ where $r = |z|$ and $\theta = \arg(z)$

We shall take the values of $\theta$, such that $-\pi < \theta \leq \pi$, called the principal argument of $z$ and is denoted by arg $z$, unless specified otherwise.

**Example:** What are the polar coordinates of the complex number $z = 2 - 2\sqrt{3}i$ ?

**Solution**: In this example, $a = 2$ and $b = -2\sqrt{3}$

$$r = \sqrt{a^2 + b^2} = \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{4 + 12} = 4$$

$$\tan\theta = \frac{b}{a} = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$$

$$\Rightarrow \theta = \frac{2\pi}{3}$$

Hence the polar coordinates of the number are $\left(4, \frac{2\pi}{3}\right)$.

**Example:** Write down the polar form of $z = -1 + i\sqrt{3}$.

**Solution:** Let's first get $r$

$$r = |z| = \sqrt{1 + 3} = 2$$

Now let's find arg $z$

$$\tan\theta = \frac{b}{a} = -\sqrt{3}$$

$$\therefore \quad \theta = (-\sqrt{3}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

So the **polar form** of $z$ is $2[\cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})] = 2[\cos(\pi - \frac{\pi}{3}) + i\sin(\pi - \frac{\pi}{3})]$

$$= 2[-\cos\frac{\pi}{3} + i\sin(\frac{\pi}{3})]$$

Having looked at a specific case, we will now look at the general case of a complex number $z$ having $(a, b)$ as the Cartesian coordinates and $(r, \theta)$ as the polar coordinates.

By Pythagoras Theorem

$$r^2 = a^2 + b^2 \Rightarrow r = \sqrt{a^2 + b^2}$$

By using the basic trigonometric ratios,

$cos\theta = \frac{a}{r}$ and $sin\theta = \frac{b}{r}$ or $rcos\theta = a \; and \; rsin\theta = b$

The cartesian form of the complex number is given by

$$z = a + ib = r(cos\theta + isin\theta)$$

## 5. Summary

- The conjugate of the complex number $z = a + ib$, denoted by $z$ , is given by

$z = a – ib$.

- The polar form of the complex number $z = x + iy$ is $r$ (cosθ + $i$ sinθ), where

$r = \sqrt{x^2 + y^2}$(the modulus of $z$) and $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{x}{r}$ (θ is known as the argument of $z$).

The value of $\theta$, such that $-\pi < \theta \leq \pi$, is called the *principal argument* of $z$.