1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 1 (Class XI, Semester - 1)
Module Name/Title	Complex Numbers – Introduction; Operations on Complex Numbers: Part 1
Module Id	kemh_10501
Pre-requisites	• The Real number system and operations within this system • Solving linear equations • Solving quadratic equations with real roots and the knowledge that imaginary roots may exist • Rules for indices and surds for example $\sqrt{xy} = \sqrt{x}.\sqrt{y}$ provided both are not negative
Objectives	After going through this lesson, the learners will be able to: **1.** understand the need of numbers beyond real numbers. **2.** understand the concept of *i* and its application **3.** define a complex number (z = a+ib) and identify its real and imaginary parts **4.** concept of purely real and purely imaginary complex number **5.** get familiar with equality of complex numbers **6.** understand the addition and subtraction of complex numbers and its properties **7.** understand the multiplication of complex numbers and its properties **8.** understand the division of complex numbers and its properties **9.** identify the multiplicative inverse or reciprocal of a complex number **10.** Solve a quadratic equation having complex root(s)
Keywords	Real Numbers; Imaginary; Complex Numbers; Quadratic Equations

2. Development Team

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1. INTRODUCTION

Consider the equation and solve it in the set of Real Numbers

$x^2 + 1 = 0$

$x^2 = -1$

$x = \pm\sqrt{-1}$

What number when multiplied by itself (squared) gives -1?

Since the square of any real number is positive, the above equation does not have any solution in the domain of Real Number system.

Therefore, it becomes necessary to extend the Real Number system to a system where $x^2 = -1$ has a solution.

We denote $\sqrt{-1}$ by the symbol *I*, read as 'iota'.

Then we have $i^2 = -1$.

This means that $i$is the solution of the equation $x^2 + 1 = 0$

COMPLEX NUMBERS (C)

A number of the form $a + ib$, where $a, b \in$R is called a Complex Number.

$i^2 = -1$ and $i = \pm\sqrt{-1}$

$$z = a + ib$$

Re (z) = a and Im (z) = b

i.e. Real part of the number z is a and the Imaginary part of the number z is b.

Two complex numbers $z_1 = a + ib$and $z_2 = c + id$are equal if $a = c$ and $b = d$.

Example: If $5x + i(2x - y) = 3 + i(-5)$, where x and y are real numbers, then find the values of x and y.

Solution: We have

$5x + i(2x - y) = 3 + i(-5)$ ... (1)

Equating the real and the imaginary parts of (1), we get

$5x = 3,\ 2x - y = -5$,

which, on solving simultaneously, give $x = \frac{3}{5}; y = \frac{31}{5}$

Power of i

While $i^2 = -1$, what would i^3, i^4, i^5, ... and higher powers of i be

Observe the following pattern

$$i^1 = i$$

$$i^2 = -1$$

$$i^3 = i^2.i = -1.i = -i$$

$$i^4 = i^2.i^2 = -1.-1 = 1$$

$$i^5 = i^4.i = i$$

$$i^6 = i^5.i = i.i = i^2 = -1$$

$$i^7 = i^3 = -i$$

$$i^8 = i^4 = 1$$

In other words, to calculate any high power of i, you can convert it to a lower power by taking the closest multiple of 4 that's no bigger than the exponent and subtracting this multiple from the exponent. Here's how the shortcut works:

In general, for any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$

$i^{99} = i^{96+3} = i^{(4\times24)+3} = i^3 = i^2.i = -1.i = -i$

Here are a few more examples:

- **Simplify i^{17}.**

$i^{17} = i^{16+1} = i^{4\cdot4+1} = i^1 = \boldsymbol{i}$

- **Simplify i^{120}.**

$i^{120} = i^{4\cdot30} = i^{4\cdot30+0} = i^0 = \mathbf{1}$

- **Simplify $i^{64,002}$.**

$i^{64,002} = i^{64,000+2} = i^{4\cdot16,000+2} = i^2 = \mathbf{-1}$

2. Algebra of Complex Numbers

Addition and Subtraction of Complex Numbers

If $z_1 = a + ib$ and $z_2 = c + id$, then

We define

$$z_1 + z_2 = (a + c) + i(b + d)$$

And $z_1 - z_2 = (a - c) + i(b - d)$

Example: Let $z = 4 + i$ and $w = 2 + 2i$. *Find* $w+z$.

Solution: $w + z$

$= (2 + 2i)+(4 + i)$

$= 6 + 3i$

$w - z$

$= (2 + 2i) - (4 + i)$

$= 2 + 2i - 4 - i$

$= -2 + i$

The addition of complex numbers satisfy the following properties:

(i) **The closure law**: The sum of two complex numbers is a complex number,

i.e., $z_1 + z_2$ is a complex number for all complex numbers z_1 *and* z_2

(ii) **The commutative law:** For any two complex numbers z_1 *and* z_2,

$$z_1 + z_2 = z_2 + z_1$$

(iii) **The associative law:** For any three complex numbers z_1, z_2, z_3

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

(iv) **The existence of additive identity:** There exists the complex number $0 + i0$ (denoted as 0), called the *additive identity* or the *zero complex number*, such that, for every complex number z, $z + 0 = z$.

(v) **The existence of additive inverse***:* To every complex number $z = a + ib$, we have the complex number $-a + i(-b)$ (denoted as $-z$),called the *additive inverse or negative of z.* We observe that $z + (-z) = 0$*(the additive identity).*

Multiplication of Complex Numbers

If $z_1 = a + ib$ and $z_2 = c + id$, then

we define

$$z_1 . z_2 = (ac - db) + i(ad + bc)$$

- **Simplify (2 –*i*)(3 + 4*i*).**

$(2 - i)(3 + 4i) = (2)(3) + (2)(4i) + (-i)(3) + (-i)(4i)$

$= 6 + 8i - 3i - 4i^2 = 6 + 5i - 4(-1)$

$= 6 + 5i + 4 =$ $\mathbf{10 + 5}\boldsymbol{i}$

The multiplication of complex numbers possesses the following properties, which we state without proofs.

(i)**The closure law** The product of two complex numbers is a complex number, the product $z_1 . z_2$is a complex number for all complex numbers z_1, z_2.

(ii) **The commutative law** For any two complex numbers z_1, z_2,$z_1 . z_2 = z_2 . z_1$

(iii) **The associative law** For any three complex numbers z_1, z_2, z_3

$$(z_1 . z_2) . z_3 = z_1 . (z_2 . z_3)$$

(iv) **The existence of multiplicative identity** There exists the complex number $1 + i0$ (denoted as 1), called the *multiplicative identity* such that $z.1 = z$, for every complex number z.

(v) **The existence of multiplicative inverse** For every non-zero complex number $z = a + ib$ or $a + bi$($a \neq 0$, $b \neq 0$), we have the complex number

$$\frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}$$

(denoted by $\frac{1}{z}$ or z^{-1}), called the *multiplicative inverse* of z such that $z.\frac{1}{z} = 1$

(vi)**The distributive law** For any three complex numbers z_1, z_2, z_3

(a) $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

(b) $(z_1 + z_2)z_3 = z_1z_3 + z_2z_3$

Conjugate and Modulus of a Complex Number

If $z = a + ib$, then its conjugate denoted by $\underline{z}$ is defined as

$$\underline{z} = a - ib$$

And its modulus denoted by $|z|$ is defined as

$$|z| = \sqrt{a^2 + b^2}$$

Example: Conjugate of $2 + 3i$ is $2 - 3i$ and its modulus is $\sqrt{4+9} = \sqrt{13}$

Division of Complex Numbers

If $z_1 = a + ib$ and $z_2 = c + id$, then

$$\frac{z_1}{z_2} = \frac{z_1.\underline{z_2}}{z_2.\underline{z_2}} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i((bc-ad)}{c^2+d^2}$$

$$= \left(\frac{ac+bd}{c^2+d^2}\right) + i\left(\frac{bc-ad}{c^2+d^2}\right)$$

Example: $\frac{3}{2+i}$

$$\frac{3}{2+i} = \frac{3}{2+i}\cdot\frac{2-i}{2-i} = \frac{3(2-i)}{(2+i)(2-i)}$$

$$= \frac{6-3i}{4-2i+2i-i^2} = \frac{6-3i}{4-(-1)}$$

$$= \frac{6-3i}{4+1} = \frac{6-3i}{5} = \frac{6}{5} - \frac{3}{5}i$$

Example: $(4+2i) \div (3-i)$

$$\frac{(4+2i)}{(3-i)} = \frac{(4+2i)}{(3-i)}\cdot\frac{(3+i)}{(3+i)}$$

$$= \frac{12 + 4i + 6i + 2i^2}{9 + 3i - 3i - i^2} = \frac{12 + 10i + 2(-1)}{9 - (-1)}$$

$$= \frac{10 + 10i}{10} = \frac{1 + i}{1} = 1 + i$$

3. Quadratic Equations

We are already familiar with the quadratic equations and have solved them in the set of real numbers in the cases where discriminant is non-negative, i.e., ≥ 0,

Let us consider the following quadratic equation:

$ax^2 + bx + c = 0$ with real coefficients a, b, c and $a \neq 0$.

Recall that the quadratic formula is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If the discriminant i.e. $\boldsymbol{b^2 - 4ac > 0}$, there are two real solutions to $\boldsymbol{ax^2 + bx + c = 0}$.
- If $\boldsymbol{b^2 - 4ac = 0}$, there are two repeated roots and hence one real solution to $\boldsymbol{ax^2 + bx + c = 0}$.
- If $b^2 - 4ac < 0$**,** there is no real solution to $\boldsymbol{ax^2 + bx + c = 0}$

But now, we know that we can find the square root of negative real numbers in the set of complex numbers. Therefore, the solutions to the above equation are available in the set of complex numbers which are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{4ac - b^2}\, i}{2a}$$

Example: Solve $x^2 + x + 1 = 0$

Solution: Here, $b^2 - 4ac = 1^2 - 4 \times 1 \times 1 = 1 - 4 = -3$

Therefore, the solutions are given by $\frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$

4. Summary

- A number of the form $a + ib$, where a and b are real numbers, is called a complex number, a is called the real part and b is called the imaginary partof the complex number.
- Let $z_1 = a + ib$ and $z_2 = c + id$. Then

(i) $z_1 + z_2 = (a + c) + i\,(b + d)$

(ii) $z_1\, z_2 = (ac - bd) + i\,(ad + bc)$

- The conjugate of the complex number $z = a + ib$, denoted by $\underline{z}$, is given by $a - ib$

- For any non-zero complex number $z = a + ib\ (a \neq 0,\ b \neq 0)$, there exists the complex number $\frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}$ (denoted by $\frac{1}{z}$ or z^{-1}), called the *multiplicative inverse* of z such that $z.\frac{1}{z} = 1$

- If $z_1 = a + ib$ and $z_2 = c + id$, then

$$\frac{z_1}{z_2} = \left(\frac{ac+bd}{c^2+d^2}\right) + i\left(\frac{bc-ad}{c^2+d^2}\right)$$

- For any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$
- The solutions of the quadratic equation $ax^2 + bx + c = 0$,

where $a, b, c \in R, a \neq 0, b^2 - 4ac < 0$, are given by

$$x = \frac{-b \pm \sqrt{4ac - b^2}\,i}{2a}$$