1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 01 (Class XI, Semester - 1)
Module Name/Title	Principle of Mathematical Induction - Part 2
Module Id	kemh_10402
Pre-requisites	Knowledge about Mathematical Inductions.
Objectives	After going through this lesson, the learners will be able to understand the following: ● Principle of Mathematical Induction ● Examples of Mathematical Induction
Keywords	Mathematical Induction, Examples

2. Development Team

Role	Name	Affiliation
National MOOC Coordinator (NMC)	Prof. Amarendra P. Behera	CIET, NCERT, New Delhi
Program Coordinator	Dr. Mohd. Mamur Ali	CIET, NCERT, New Delhi
Course Coordinator (CC) / PI	Dr. Til Prasad Sarma	DESM, NCERT, New Delhi
Course Co-Coordinator / Co-PI	Dr. Mohd. Mamur Ali	CIET, NCERT, New Delhi
Subject Matter Expert (SME)	Mr. Rahul Sofat	Air Force Golden Jubilee Institute, New Delhi
Review Team	Prof. Bhim Prakash Sarrah	Assam University, Tezpur

Example 1: Prove by mathematical induction $2n+1 < 2^n$, for all natural numbers $n \geq 3$.

Solution:

Let P(n) be the given statement, i.e.,

P(n): $2n+1 < 2^n$ for $n \geq 3$

For n = 3, P(3): $2\times3+1 < 2^3$

i.e., $7 < 8$, which is true

Assume that P(n) is true for some positive integer k, i.e.,

P(k) : $(2k+1) < 2^k$ + ___________(1)

We shall now prove that P(k+1) is true, i.e.,

P(k+1): $2(k+1) +1 < 2^{k+1}$ __________(2)

Now, $2(k+1)+1= 2k+3$

$= (2k+1)+2$

$< 2^k + 2 \leq 2(2^k) = 2^{k+1}$

Hence, $2(k+1) + 1< 2^{k+1}$

Thus, P(k+1) is true whenever P(k) is true.

Hence, from the principle of mathematical induction, the statement P(n) is true for all n.

Example 2: Prove by induction, that $x^{2n} – y^{2n}$ is divisible by (x+y) where x,y are distinct real numbers for all $n \in N$.

Solution:

Let P(n) be the given statement, i.e.,

P(n): $x^{2n} – y^{2n}$ is divisible by (x+y)

P(1): **$x^2 – y^2 = (x-y)(x+y)$ which is divisible by (x+y)**

Hence P(1) is true.

Assume that P(k) is true, i.e.,

P(k): $x^{2k} – y^{2k}$ is divisible by (x+y)_____________(1)

We shall now prove that P(k+1) is true, i.e.,

P(k+1): $x^{2(k+1)} – y^{2(k+1)}$ is divisible by (x+y)__________(2)

Now, let us consider the expression: $x^{2k} x^2 – y^{2k} y^2$

$= x^{2k} x^2 – y^{2k}x^2 + y^{2k} x^2 – y^{2k} y^2$

$= x^2 (x^{2k} - y^{2k}) + y^{2k}(x-y)(x+y)$

Since, the expression $x^2 (x^{2k} - y^{2k})$ is divisible by (x+y) [using (1)] and $y^{2k}(x-y)(x+y)$ is divisible by (x+y), therefore, P(k+1) is true whenever P(k) is true

Hence, by principle of mathematical induction, P(n) is true for all n ϵ N.

Example 3: By mathematical induction, prove that $\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}$ **is a positive integer for all n ϵ N.**

Solution:

Let P(n) be the statement: $\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}$ is a positive integer

P(1): $\frac{1^5}{5}+\frac{1^3}{3}+\frac{7\times 1}{15}=\frac{1}{5}+\frac{1}{3}+\frac{7}{15}$ = 1 which is a positive integer

Hence P(1) is true.

Assume that P(k) is true i.e.,

$P(k)=\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}$ is a positive integer _______(1)

We shall now prove that P(k+1) is true, i.e.,

$P(k)=\frac{(K+1)^5}{5}+\frac{(K+1)^3}{3}+\frac{7(K+1)}{15}$ is a positive integer.

Now, let us consider the expression, $\frac{(K+1)^5}{5}+\frac{(K+1)^3}{3}+\frac{7(K+1)}{15}$

$$= \frac{k^5+5k^4+10k^3+10k^2+5k+1}{5}+\frac{k^3+3k^2+3k+1}{3}+\frac{7k+7}{15}$$

$$= \left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)+\left(k^4+2k^3+2k^2+k+k^2+k\right)+\left(\frac{1}{5}+\frac{1}{3}+\frac{7}{15}\right)$$

= a positive integer + a positive integer +1 [using (1)]

= a positive integer i.e., P(k+1) is true whenever P(k) is true

Hence, by principle of mathematical induction, P(n) is true ∀ n ϵ N.

Example 4: Prove by the principle of mathematical induction, ∀ n ϵ N.

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots\cdots\cdots\cdots\cdots\cdots+\frac{1}{1+2+3+\cdots\cdots+n}=\frac{2n}{n+1}$$

Solution:

$$P(n)=1+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots\cdots\cdots\cdots\cdots\cdots+\frac{1}{1+2+3+\cdots\cdots+n}=\frac{2n}{n+1}$$

P(n) = $\frac{2\times 1}{1+1}=1, \textit{which is true}.$

Assume that P(k) is true, i.e.,

$$¿P(k)=1+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots\cdots\cdots\cdots\cdots\cdots+\frac{1}{1+2+3+\cdots\cdots+k}=\frac{2k}{k+1} \quad (1)$$

We shall now prove P(k+1) is true, i.e.,

$$P(k+1)=1+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots\cdots\cdots\cdots\cdots\cdots+\frac{1}{1+2+3+\cdots\cdots+k}$$

$$\frac{+1}{1+2+3+\cdots\cdots+k+(k+1)}=\frac{2k}{k+1}+\frac{1}{\frac{(k+1)(k+2)}{2}}\,[using(1)]$$

$$=\frac{2}{k+1}\left[k+\frac{1}{k+2}\right]=\frac{2}{k+1}\left[\frac{k^2+2k+1}{k+2}\right]$$

$$=\frac{2}{k+1}\,\frac{(k+1)^2}{k+2}=\frac{2(k+1)}{k+2}=RHS$$

Thus, P(k+1) is true whenever P(k) is true.

Hence, by principle of mathematical induction, P(k) is true.

Example 5: prove by the principle of mathematical induction, ∀ n ϵ N

$$\sin\sin\theta+\sin\sin 2\theta+\sin\sin 3\theta+\cdots\cdots+\sin\sin n\theta = \frac{\sin\sin\left(\frac{n+1}{2}\right)\theta\sin\sin\left(\frac{n}{2}\right)\theta}{\sin\sin\frac{\theta}{2}}$$

Solution:

Let P(n) be the given statement i.e.,

P(n): $\sin\sin\theta+\sin\sin 2\theta+\sin\sin 3\theta+\cdots\cdots+\sin\sin n\theta \quad = \quad \frac{\sin\sin\left(\frac{n+1}{2}\right)\theta\sin\sin\left(\frac{n}{2}\right)\theta}{\sin\sin\frac{\theta}{2}}$

P(1): $\sin\sin\theta=\frac{\sin\sin\left(\frac{1+1}{2}\right)\theta\sin\sin\left(\frac{1}{2}\right)\theta}{\sin\sin\frac{\theta}{2}}=\sin\sin\theta$, which is true

Assume that P(k) is true i.e.,

P(n): $\sin\sin\theta+\sin\sin 2\theta+\sin\sin 3\theta+\cdots\cdots+\sin\sin k\theta \quad = \quad \frac{\sin\sin\left(\frac{k+1}{2}\right)\theta\sin\sin\left(\frac{k}{2}\right)\theta}{\sin\sin\frac{\theta}{2}}$

____________(1)

We shall now prove P(k+1) is true, i.e.,

P(n): $\sin\sin\theta+\sin\sin 2\theta+\sin\sin 3\theta+\cdots\cdots+\sin\sin k\theta+\sin\sin(k+1)\theta \quad =$

$\frac{\sin\sin\left(\frac{k+2}{2}\right)\theta\sin\sin\left(\frac{k+1}{2}\right)\theta}{\sin\sin\frac{\theta}{2}}$ ___(2)

Now let us consider

$\theta+\sin\sin 2\theta+\sin\sin 3\theta+\cdots\cdots+\sin\sin k\theta+\sin\sin(k+1)\theta$

$= \frac{\sin\sin\left(\frac{k+1}{2}\right)\theta\sin\sin\left(\frac{k}{2}\right)\theta}{\sin\sin\frac{\theta}{2}} + 2\sin\left(\frac{k+1}{2}\right)\theta\cos\left(\frac{k+1}{2}\right)\theta$ [using (1)]

$= \frac{\sin\sin\left(\frac{k+1}{2}\right)\theta}{\sin\sin\frac{\theta}{2}}\left[\sin\sin\left(\frac{k}{2}\right)\theta+2\sin\sin\frac{\theta}{2}\cos\left(\frac{k+1}{2}\right)\theta\right]$

$= \frac{\sin\sin\left(\frac{k+1}{2}\right)\theta}{\sin\sin\frac{\theta}{2}}\left[\sin\sin\left(\frac{k}{2}\right)\theta+\frac{k\theta}{2}\right]$

$$= \frac{\sin\sin\left(\frac{k+1}{2}\right)\theta\sin\left(\frac{k+2}{2}\right)\theta}{\sin\sin\frac{\theta}{2}} = \text{RHS}$$

Thus, P(k+1) is true whenever P(k) is true.

Hence, by principle of mathematical induction, P(n) is true ∀ n ϵ N.

Example 6: Use mathematical induction to prove

$\lfloor n < \left(\frac{n+1}{2}\right)^n$ **, where n ϵ N and n >1**

Solution:

Let P(n) be the given statement, i.e.,

P(n): $\lfloor n < \left(\frac{n+1}{2}\right)^n$

Now, P(2): $\lfloor 2 < \left(\frac{2+1}{2}\right)^2$ i.e., $2 < \frac{9}{4}$ which is true.

Assume that P(k) is true i.e.,

P(n): $\lfloor k < \left(\frac{k+1}{2}\right)^k$ ____________(1)

We shall now prove P(k+1) is true, i.e.,

P(k): $\lfloor k+1 < \left(\frac{k+2}{2}\right)^{k+1}$ ____________(2)

Now, $\lfloor k+1 = (k+1)\lfloor k < (k+1)\left(\frac{k+1}{2}\right)^k = \frac{(k+1)^{k+1}}{2^k}$

$\lfloor k+1 < \frac{(k+1)^{k+1}}{2^k}$ ______ (3)

By binomial theorem,

$$\left(1+\frac{1}{k+1}\right)^{k+1} = 1 + (k+1)\frac{1}{k+1} + \cdots\cdots$$

$$\Longrightarrow \left(\frac{k+2}{k+1}\right)^{k+1} > 2$$

$$\Longrightarrow \frac{(k+2)^{k+1}}{(k+1)^{k+1}} > \frac{2 \times 2^k}{2^k}$$

$$\Longrightarrow \frac{(k+2)^{k+1}}{2^{k+1}} > \frac{(k+1)^{k+1}}{2^k}$$

$$\Longrightarrow \frac{(k+1)^{k+1}}{2^k} < \left(\frac{k+2}{2}\right)^{k+1} \quad \_\_\_\_\_\_\_\_\_\_(4)$$

From (3) and (4)

$$\lfloor k+1 < \left(\frac{k+2}{2}\right)^{k+1}$$

Thus, P(k+1) is true whenever P(k) is true.

Hence, by principle of mathematical induction P(k) is true.