1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 01 (Class XI, Semester - 1)
Module Name/Title	Principle of Mathematical Induction- Part 1
Module Id	kemh_10401
Pre-requisites	Knowledge about Mathematical Inductions.
Objectives	After going through this lesson, the learners will be able to understand the following: • Principle of Mathematical Induction • Examples of Mathematical Induction
Keywords	Mathematical Induction, Examples

2. Development Team

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Introduction: In this module, we will prove certain results or statements (including inequality) in terms of n. where n is a positive integer using principle of mathematical induction.

The principle of mathematical induction

Suppose there is a given statement P(n) involving the natural number n such that

1. The statement is true for n=1, i.e., P(1) is true and
2. If the statement is true for n = k(where k is some positive integer), then the statement is also true for n = k+1, i.e., truth of P(k) implies the truth of P(k+1).

   Then, P(n) is true for all natural number n.

Note:

- In the above statement property (i) is simply a statement of fact. There may be instances when a statement is true for n ≥ r, where r is some positive integer other than 1 (say 2 or 3 etc.)

  We shall verify the result for n = r i.e.,in step 1, we shall show P(r) is true.
- The property given in step (ii) is a conditional property. It does not assert that the given statement is true for n = k, but only that if it is true for n = k, then it is true for n = k+1. This step is also referred as inductive step.

Example-1 For all n≥1, prove that

$$1+2+3+\ldots+\frac{n(n+1)}{2}$$

Solution: let the given statement be P(n), i.e.,

P (n): $1+2+3+\ldots+\frac{n(n+1)}{2}$

For n = 1, P (1): 1 = 1(1+1)/2 = 1, which is true

Assume that P (n) is true for some positive integer k, i.e.,

P (k+): 1+2+3+……+k = $\frac{k(k+1)}{2}$ ……. (1)

We shall now prove that P (k+1) is also true, i.e.

P (k+1): 1+2+3+……+k + (k+1) = $k+1+1(k+1)$ ………. (2)

LHS = (1+2+3+…. +k) + (k+1)

$$= \frac{k(k+1)}{2} + (k+1) \qquad \text{using (1)}$$

$$= (k+1) \left[\frac{R}{2} + 1\right]$$

$$= \frac{(R+1)(R+2)}{2}$$

$$= \frac{(k+1)\{(k+1)+1\}}{2}$$

= RHS

Thus P(R+1) is true, whenever P (k) is true. Hence, from the principle of mathematical induction, the statement P(n) is true for all natural number n

Example-2 Prove the following by using the principle of mathematical induction for all n ∈ N: $a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$

Solution: let the given statement be P (n), i.e.,

P (n): $a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$

For n = 1, P (1): a = $\frac{a(r-1)}{(r-1)}$ = a , which is true

Assume that P (n) is true for some positive integer R, i.e.

P (n): $a + ar + ar^2 + \ldots + ar^{R-1} = \frac{a(r^R - 1)}{r - 1} \ldots$ (1)

We shall now prove that P(R+1) is also true, i.e.

P(R+1): $a + ar + ar^2 + \ldots + ar^{R-1} + ar^R = \frac{a(r^R - 1)}{r - 1}$ … (2)

LHS = $(a + ar + ar^2 + \ldots + ar^{R-1}) + ar^R$

$$= \frac{a(r^R - 1)}{r - 1} + ar^R \qquad \text{(using 1)}$$

$$= \frac{ar^R - a + ar^R(r-1)}{r - 1}$$

$$= \frac{ar^R - a + ar^{R+1} - ar^R}{r - 1}$$

$$= \frac{a\left(r^{R+1}-1\right)}{r-1} = \text{RHS}$$

Thus P (R+1) is true, whenever P (k) is true.

Hence from the principle of mathematical induction, the statement P (n) is true for all natural numbers n.

Example-3 Using the principle of mathematical induction

Prove that: $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

Where n is any positive integer

Solution: we can write

P (n): $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

We note that P (1): $\frac{1}{1.3}=\frac{1}{2\times 1+1}=\frac{1}{3}$, which is true

Thus P (n) is true for n = 1

Assume P (n) is true for some natural number R,

i.e. P(R) : $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2R-1)(2R+1)}=\frac{R}{2R+1}$ …(1)

we need to prove P(R+1) is true i.e.

P(R+1): $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2R-1)(2R+1)}+\frac{1}{(2R+1)(2R+3)}=\frac{R+1}{2(R+1)+1}$ …(2)

LHS = $\left[\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2R-1)(2R+1)}\right] + \frac{1}{(2R+1)(2R+3)}$ (using 1)

$$= \frac{R}{2R+1} + \frac{1}{(2R+1)(2R+3)}$$

$$= \frac{1}{2R+1}\left[R+\frac{1}{(2R+3)}\right]$$

$$= \frac{1}{2R+1}\frac{R(2R+3)+1}{2R+3}$$

$$= \frac{1}{2R+1}\times\frac{2R^2+3R+1}{2R+3}$$

$$= \frac{1}{2R+1}\times\frac{(R+1)(2R+1)}{2R+3}$$

$$= \frac{R+1}{2R+3}$$

$$= \frac{R+1}{(2R+2)+1}$$

$$= \frac{R+1}{(2R+2)+1}$$ = RHS

Thus P(R+1) is true whenever P(R) is true.

Hence, by the principle of mathematical induction, P(n) is true for all natural number N

Example-4: Prove by the principle of Mathematical Induction

$$1^3+2^3+3^3+\ldots+n^3=(1+2+3+\ldots+n)^2$$

Solution: let the given statement be P(n), i.e.,

P(n): $1^3+2^3+3^3+\ldots+n^3=(1+2+3+\ldots+n)^2$

For n=1, P(1): 1^3 $=1=1^2$, which is true

Assume that P(n) is true for some positive integer k, i.e.,

P(k): $1^3+2^3+3^3+\ldots+k^3=(1+2+3+\ldots+k)^2$ … (1)

We shall now prove that P(k+1) is also true, i.e.,

P(k+1): $1^3+2^3+3^3+\ldots+k^3+(k+1)^3=(1+2+3+\ldots+(k+1))^2$ … (2)

LHS = $1^3+2^3+3^3+\ldots+k^3$)+ $(k+1)^3$

$= (1+2+3+\ldots+k)^2 + (k+1)^3$

$= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$

$= \frac{k^2(k+1)^2}{4}$ + + $(k+1)^3$

$= (k+1)^2\left[\frac{k^2}{4}+k+1\right]$

$= \quad (k+1)^2\left[\frac{k^2+4k+4}{4}\right]$

$= \quad \left[\frac{(k+1)^2(k+2)^2}{2}\right]^2$

$= \quad (1+2+3+\ldots+(k+1))^2$ = RHS

Thus P(k+1) is true, whenever P(k) is true.

Hence, from the principle of mathematical induction, the statement P(n) is true for all natural numbers n.

Example: 5 prove by the Principle of mathematical induction for all n∈N

$1.2+2 2^2+3 2^3+\ldots+n 2^n=(n-1)2^{n+1}+2$

Solution: let the given statement be P(n), i.e.

$P(n):1.2+2 2^2+3 2^3+\ldots+n 2^n=(n-1)2^{n+1}+2$

For n = 1, P(1) : 1.2 = 2 = (1-1) 2^2 +2 = 2, which is true

Assume that P(n) is true for some positive integer k, i.e.,

$P(k):1.2+2 2^2+3 2^3+\ldots+k 2^k=(k-1)2^{k+1}+2\ldots(1)$

We shall now prove that P(k+1) is also true, i.e.,

$P(k+1):1.2+2 2^2+3 2^3+\ldots+k 2^k+(k+1)2^{k+1}=(k)2^{k+2}+2\ldots(2)$

L H S = $(1.2+2 2^2+3 2^3+\ldots+k 2^k)+(k+1)2^{k+1}$

$= \left[(k-1)2^{k+1}+2\right] + (k+1)2^{k+1} \qquad [Using(1)]$

$= \quad (k-1)2^{k+1} + (k+1)2^{k+1} +2$

$= \quad (k-1+k+1)2^{k+1}+2$

= 2k 2^{k+1} + 2

= k 2^{k+2} + 2

=R H S

Thus P (k+1) is true, whenever P(k) is true.

Hence from the principle of mathematical induction, the statement P(n) is true for all natural numbers n.

Example-6: prove by the principle of mathematical induction that

7+77+777+…+ 7777…7 = $\frac{7}{81}\left(10^{n+1}-9n-10\right)$

For all n ∈ N

Solution: let the given statement be P (n), i.e.,

For n = 1, P(1): $\frac{7}{81}\left(10^{2}-9-10\right)=\frac{7}{81}\times 81=7$, which is true.

Assume that P (n) is true for some positive integer k, i.e.

7+77+777+…+ 7777…7 = $\frac{7}{81}\left(10^{k+1}-9k-10\right)\ldots(1)$

We shall now prove that P(k+1) is also true, i.e.,

P (k+1): 7+77+777+…+ 7777…7 + 77777…7 = $\frac{7}{81}\left(10^{k+2}-9(k+1)-10\right)\ldots(2)$

L H S = (7+77+777+…+ 7777…7) + 77777…7

= $\frac{7}{81}\left(10^{k+1}-9k-10\right)$ + 77777…7 (using 1)

= $\frac{7}{81}\left(10^{k+1}-9k-10\right)$ + $\left(1+10+10^{2}+\ldots+10^{k}\right)$

= $\frac{7}{81}\left(10^{k+1}-9k-10\right)$ + 7 $\frac{\left(10^{k+1}-1\right)}{10-1}$

= $\frac{7}{81}\left(10^{k+1}-9k-10+9.\,10^{k+1}-9\right)$

= $\frac{7}{81}\left(10^{k+2}-9(k+1)-10\right)$ = R H S

Hence P (k+1) is true, whenever P (k) is true

Hence from the principle of mathematical induction the statement P(n) is true for all natural numbers n.

Example-7: Using principle of mathematical induction, prove that

$$\cos\cos\alpha\cos\cos 2\alpha\cos\cos 3\alpha\cos\cos 4\alpha\ldots.\cos\cos\left(2^n\alpha\right)=\frac{\sin\sin\left(2^n\alpha\right)}{2^n\sin\sin\alpha}$$

For all n ∈ N

Solution: let the given statement be P (n), i.e.

P (n): $\cos\cos\alpha\cos\cos 2\alpha\cos\cos 3\alpha\cos\cos 4\alpha\ldots.\cos\cos\left(2^n\alpha\right)=\frac{\sin\sin\left(2^n\alpha\right)}{2^n\sin\sin\alpha}$

For n = 1, P (1): $\cos\cos\alpha = \frac{\sin\sin\left(2\alpha\right)}{2\sin\sin\alpha} = \frac{\left(\alpha\right)\cos\cos\alpha}{2\sin\sin\alpha} = \cos\cos\alpha$,

Assume that P (n) is true for some positive integer k, i.e.

P (k): $\cos\cos\alpha\cos\cos 2\alpha\cos\cos 3\alpha\cos\cos 4\alpha\ldots.\cos\cos\left(2^k\alpha\right)=\frac{\sin\sin\left(2^k\alpha\right)}{2^k\sin\sin\alpha}\ldots.\left(1\right)$

Let us now prove that P (k+1) is true, i.e.

P (k + 1):

$$\cos\cos\alpha\cos\cos 2\alpha\cos\cos 3\alpha\cos\cos 4\alpha\ldots.\cos\cos\left(2^{k-1}\alpha\right)\cos\cos\left(2^k\alpha\right)=\frac{\sin\sin\left(2^{k+1}\alpha\right)}{2^{k+1}\sin\sin\alpha}\ldots.\left(2\right)$$

L H S = $\left[\cos\cos\alpha\cos\cos 2\alpha\cos\cos 3\alpha\cos\cos 4\alpha\ldots.\cos\cos\left(2^{k-1}\alpha\right)\right]\cos\cos\left(2^k\alpha\right)$

$= \left[\frac{\sin\sin\left(2^k\alpha\right)}{2^k\sin\sin\alpha}\right]\cos\cos\left(2^k\alpha\right)$

$= \frac{2\sin\sin\left(2^k\alpha\right)\cos\cos\left(2^k\alpha\right)}{2^{k+1}\sin\sin\alpha}$

$= \frac{\sin\sin\left(2^{k+1}\alpha\right)}{2^{k+1}\sin\sin\alpha}$ = R H S

Hence P (k+1) is true, whenever P (k) is true.

Hence from the principle of mathematical induction the statement P (n) is true for all natural numbers n.