1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics (Class XI, Semester - 1)
Module Name/Title	Trigonometry: Part 8
Module Id	kemh_10308
Pre-requisites	Knowledge about Trigonometric Functions.
Objectives	After going through this lesson, the learners will be able to understand the following: 1.Introduction 2. Sine Rule 3. Cosine Rule 4.Summary
Keywords	Acute angled triangle, Obtuse angled triangle, Right angled triangle, sine rule, cosine rule.

2. Development Team

Role	Name	Affiliation
National MOOC Coordinator (NMC)	Prof. Amarendra P. Behera	CIET, NCERT, New Delhi
Program Coordinator	Dr. Indu Kumar	CIET, NCERT, New Delhi
Course Coordinator (CC) / PI	Prof. Til Prasad Sarma	DESM, NCERT, New Delhi
Course Co-Coordinator	Ms. Anjali Khurana	CIET, NCERT, New Delhi
Subject Matter Expert (SME)	Ms. Kiran Seth	HOD (Math), Amity International School, Saket, New Delhi.
Review Team	Dr. S.K.S. Gautum	Retd. Professor DESM, NCERT, New Delhi

TABLE OF CONTENTS

1. Introduction

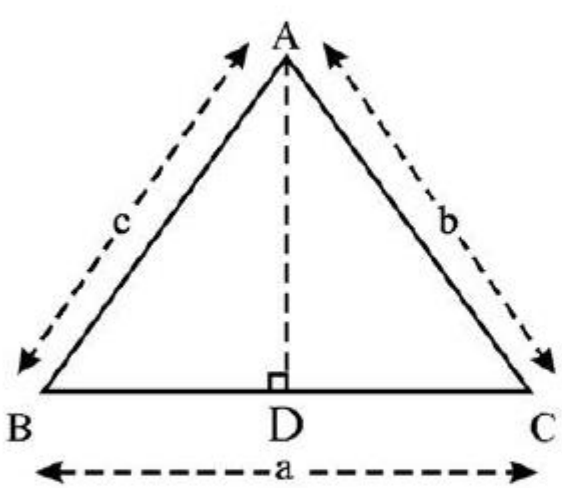
In this module we will discuss sine and cosine function formulae their derivations and their application.

2. Sine Rule

The sine rule states that the lengths of the sides of a triangle are proportional to the sines of angles opposite to them i.e. in $\Delta ABC, \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Proof: Three cases arise

Case (i) When Δ ABC is acute angled triangle



Let a = BC, b = AC and c = AB

From vertex A, draw AD⊥BC

$$\text{In } \Delta ABD, \frac{AD}{AB} = \sin B \Rightarrow AD = c \sin B \quad \text{..... (i)}$$

$$\text{In } \Delta ACD, \frac{AD}{AC} = \sin C \Rightarrow AD = b \sin C \quad \text{........ (ii)}$$

From (i) and (ii) we get, c sin B = b sin C

$$\text{or } \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{........... (A)}$$

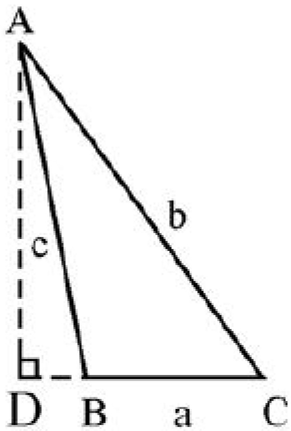
Similarly, by drawing BE⊥AC, we can prove that

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \text{........... (B)}$$

From (A) and (B), we see that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Case (ii) When Δ ABC is obtuse angled triangle.



From vertex A, draw $AD \perp BC$ produced.

In Δ ABC,

$$\frac{AD}{AB} = \sin(180^0 - B)$$
$$\Rightarrow \frac{AD}{AB} = \sin B$$
$$\Rightarrow AD = c\sin B........(i)$$

Similarly, in ΔACD, $\frac{AD}{AC} = \sin C$

or AD = b sin C .... (ii)

From (i) and (ii), we get

$$c \sin B = b \sin C \text{ or } \frac{b}{\sin B} = \frac{c}{\sin C}$$

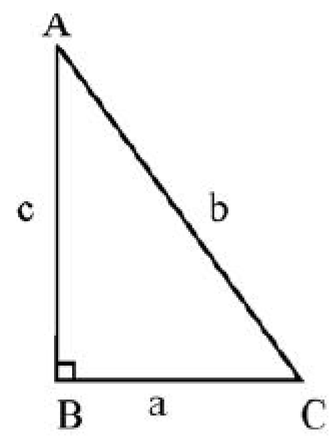
Similarly, by drawing BE⊥AC, we can show that

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Hence, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Case (iii) When Δ ABC is right angled triangle.

In Δ ABC, right angled at B.



(i) $\dfrac{AB}{AC} = \sin C$ or $\dfrac{c}{b} = \sin C \Rightarrow b = \dfrac{c}{\sin C}$

(ii) $\dfrac{BC}{AC} = \sin A$ or $\dfrac{a}{b} = \sin A \Rightarrow b = \dfrac{a}{\sin A}$

(iii) $\sin B = \sin\dfrac{\pi}{2} = 1 \Rightarrow \dfrac{b}{\sin B} = b$

From (i), (ii) and (iii), we get

$$b = \frac{a}{\sin A} = \frac{c}{\sin c} = \frac{b}{\sin B}$$

$$\Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$\therefore$ From all the three cases, we see that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Note : (i) $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$

$\Rightarrow a = k \sin A,\ b = k \sin B,\ c = k \sin C$

(ii) $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \lambda$

$\Rightarrow \sin A = a\lambda,\ \sin B = b\lambda,\ \sin C = c\lambda$

Example 1 : In ΔABC, if $a = 2$, $b = 3$ and $\sin A = \frac{2}{3}$, find $\angle B$.

Solution : We know that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Here $a = 2$, $b = 3$ and $\sin A = \frac{2}{3}$

$$\frac{2}{\frac{2}{3}} = \frac{3}{\sin B} \Rightarrow \sin B = 1 \Rightarrow B = \frac{\pi}{2} \text{ or } 90^\circ$$

Example 2: In any triangle, prove that

(i) $\dfrac{a^2 - c^2}{b^2} = \dfrac{\sin(A-C)}{\sin(A+C)}$ (ii) $b \cos B + c \cos C = a \cos (B-C)$

Solution:

(i) $\text{LHS} = \dfrac{a^2 - c^2}{b^2} = \dfrac{k^2\sin^2 A - k^2\sin^2 C}{k^2\sin^2 B}$ [by sine formula]

$$= \frac{\sin^2 A - \sin^2 C}{\sin^2 B} = \frac{\sin(A+C)\,.\,\sin(A-C)}{\sin^2\left[180^\circ - (A+C)\right]}$$

$$= \frac{\sin(A+C)\,.\,\sin(A-C)}{\sin(A+C)\,.\,\sin(A+C)} = \frac{\sin(A-C)}{\sin(A+C)} = \text{RHS}$$

(ii) $\text{LHS} = b \cos B + c \cos C$

$= k\,[\sin B \cos B + \sin C \cos C]$

$= \dfrac{k}{2}\,[\sin 2B + \sin 2C]$

$= \dfrac{k}{2}\,[2 \sin (B+C) \cos (B-C)]$

$= k\,[\sin (180^\circ - A) \cos (B-C)]$

$= k \sin A \cos (B-C)$

$= a \cos (B-C) = \text{RHS}$

3. Cosine Rule

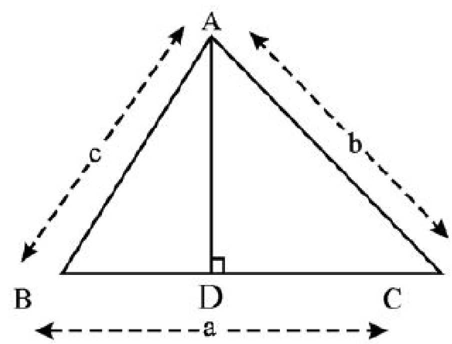
In any triangle ABC, we have

(i) $a^2 = b^2 + c^2 - 2bc\cos A$ or $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

(ii) $b^2 = a^2 + c^2 - 2ac\cos B$ or $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

(ii) $c^2 = a^2 + b^2 - 2ab\cos C$ or $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Proof : Three cases arise :



Case I : When the ΔABC is an acute angled triangle.

From vertex A, draw $AD \perp BC$

In ΔABD, $\cos B = \dfrac{BD}{c} \Rightarrow BD = c\cos B$

In ΔACD, $\cos C = \dfrac{CD}{b} \Rightarrow CD = b\cos C$

Also, $AC^2 = CD^2 + AD^2$

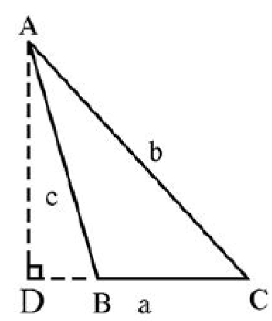
$= AD^2 + (BC - BD)^2$

$= BC^2 + (AD^2 + BD^2) - 2BC.BD$

$AC^2 = BC^2 + AB^2 - 2BC.BD$

or, $b^2 = a^2 + c^2 - 2a.c\cos B$

or, $\cos B = \dfrac{a^2 + c^2 - b^2}{2\,ac}$



Case II : When ΔABC is an obtuse angled triangle.

From vertex A, draw $AD \perp CB$ produced

In ΔABD,

$\dfrac{BD}{c} = \cos(180^0 - B) = -\cos B$

$\Rightarrow BD = -c\cos B$

Also, $AC^2 = AD^2 + CD^2$

$$= AD^2 + (BC + BD)^2$$

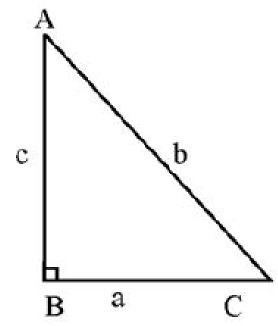
$$= AD^2 + BD^2 + BC^2 + 2BC.BD$$

$$AC^2 = AB^2 + BC^2 + 2BC.BD$$

$$\text{or } b^2 = c^2 + a^2 + 2a(-c\cos B)$$

$$\text{or } \cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

Case III : When ΔABC is a right triangle.



$b^2 = c^2 + a^2$

As $B = \frac{\pi}{2} \Rightarrow \cos B = 0$

$\therefore\ b^2 = c^2 + a^2 - 2ac\cos B$ $\quad [\because \cos B = 0]$

$\Rightarrow \cos B = \frac{c^2 + a^2 - b^2}{2ac}$

Thus, in all the three cases $\cos B = \frac{c^2 + a^2 - b^2}{2ac}$

By following the same procedure, we can prove that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \text{ and } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example 3:

In a ΔABC, prove that $a(b\cos C - c\cos B) = b^2 - c^2$

Solution:

LHS : $a(b\cos C - c\cos B)$

$$= ab\left[\frac{a^2+b^2-c^2}{2ab}\right] - ac\left[\frac{a^2+c^2-b^2}{2ac}\right]$$

$$= \frac{1}{2}\left[\cancel{a^2} + b^2 - c^2 - \cancel{a^2} - c^2 + b^2\right]$$

$$= \frac{1}{2}\left[2b^2 - 2c^2\right] = b^2 - c^2 = \text{RHS}$$

Example 4:

In a Δ ABC , prove that $\frac{c-b\cos A}{b-c\cos A} = \frac{\cos B}{\cos C}$

Solution:

$$\text{LHS} = \frac{c-b\cos A}{b-c\cos A}$$

$$= \frac{c-b\,\frac{\left(b^2+c^2-a^2\right)}{2bc}}{b-c\,\frac{\left(b^2+c^2-a^2\right)}{2bc}} = \frac{b}{c}\left[\frac{c^2+a^2-b^2}{b^2+a^2-c^2}\right]$$

$$\text{RHS} = \frac{\cos B}{\cos C} = \frac{\cancel{2}\,\cancel{a}\,b\,\left(a^2+c^2-b^2\right)}{\cancel{2}\,\cancel{a}\,c\,\left(a^2+b^2-c^2\right)} = \frac{b}{c}\left[\frac{c^2+a^2-b^2}{a^2+b^2-c^2}\right]$$

$$= \text{LHS}$$

Example 5 :

In any Δ ABC , prove that

$$2\,(bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$$

Solution:

$$\text{LHS} = 2\left\{bc\,\frac{(b^2+c^2-a^2)}{2bc} + ca\,\frac{a^2+c^2-b^2}{2ac} + ab\,\frac{a^2+b^2-c^2}{2ab}\right\}$$

$$= b^2 + c^2 - \cancel{a^2} + \cancel{a^2} + \cancel{c^2} - \cancel{b^2} + a^2 + \cancel{b^2} - \cancel{c^2}$$

$$= a^2 + b^2 + c^2 = \text{RHS}$$

Example 6:

In any ΔABC, prove that

$$\frac{b^2-c^2}{a^2}.\sin 2A + \frac{c^2-a^2}{b^2}.\sin 2B + \frac{a^2-b^2}{c^2}.\sin 2C = 0$$

Solution:

$$\text{LHS} = \frac{b^2-c^2}{a^2}.2\,(ka)\left[\frac{b^2+c^2-a^2}{2bc}\right] + \frac{c^2-a^2}{b^2}.2\,(kb)\left[\frac{a^2+c^2-b^2}{2ac}\right] + \frac{a^2-b^2}{c^2}.2\,(kc)\left[\frac{a^2+b^2-c^2}{2ab}\right]$$

$$= \frac{k}{abc}\left[(b^2-c^2)(b^2+c^2) - a^2(b^2-c^2) + (c^2-a^2)(c^2+a^2) - b^2(c^2-a^2) + (a^2+b^2)(a^2-b^2) - c^2(a^2-b^2)\right]$$

$$= \frac{k}{abc}\left[\cancel{b^4} - \cancel{c^4} - \cancel{a^2b^2} + \cancel{a^2}c^2 + \cancel{c^4} - \cancel{a^4} - \cancel{b^2}\,\cancel{c^2} + \cancel{a^2b^2} + \cancel{a^4} - \cancel{b^4} - \cancel{a^2}c^2 + \cancel{b^2c^2}\right] = 0 = \text{RHS}$$

4. Summary

In this module sine and cosine rule's derivations were taken. These rules were applied in solving various problems.