1. **Details of Module and its structure**

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics (Class XI, Semester - 1)
Module Name/Title	Trigonometry: Part 5
Module Id	kemh_10305
Pre-requisites	Knowledge about Trigonometric Functions.
Objectives	After going through this lesson, the learners will be able to understand the following: 1.Introduction 2.Trigonometric functions of multiple and submultiple angles. 3.Summary
Keywords	Multiple of an angle, Submultiple of an angle.

2. **Development Team**

Role	**Name**	**Affiliation**
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1. Introduction

In this module we will discuss the derivation trigonometric functions of multiple and submultiple angles. Also we will use these formulae to solve various problems.

2. Trigonometric functions of multiple and submultiple angles.

1. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$

$$= 1 - 2\sin^2 x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

We know that

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

Replacing y by x, we get

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1$$

Again, $\cos 2x = \cos^2 x - \sin^2 x$

$$= 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x.$$

We have $\cos 2x = \cos^2 x - \sin^2 x = \dfrac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x}$

Dividing numerator and denominator by $\cos^2 x$, we get

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}, \quad x \neq n\pi + \frac{\pi}{2}, \text{ where } n \text{ is an integer}$$

2.

$$\mathbf{\sin 2x = 2 \sin x \cos x = \frac{2\tan x}{1+\tan^2 x}} \quad x \neq n\pi + \frac{\pi}{2}, \text{ where n is an integer}$$

We have

$$\sin (x + y) = \sin x \cos y + \cos x \sin y$$

Replacing y by x, we get $\sin 2x = 2 \sin x \cos x$.

Again $\sin 2x = \dfrac{2\sin x \cos x}{\cos^2 x + \sin^2 x}$

Dividing each term by $\cos^2 x$, we get

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

3.

$$\mathbf{\tan 2x = \frac{2\tan x}{1-\tan^2 x}} \quad \text{if } 2x \neq n\pi + \frac{\pi}{2}, \text{ where n is an integer}$$

We know that

$$\tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Replacing y by x, we get $\tan 2x = \dfrac{2 \tan x}{1-\tan^2 x}$

4.

$$\mathbf{\sin 3x = 3 \sin x - 4 \sin^3 x}$$

We have,

$$\begin{aligned}
\sin 3x &= \sin (2x + x) \\
&= \sin 2x \cos x + \cos 2x \sin x \\
&= 2 \sin x \cos x \cos x + (1 - 2\sin^2 x) \sin x \\
&= 2 \sin x (1 - \sin^2 x) + \sin x - 2 \sin^3 x \\
&= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x \\
&= 3 \sin x - 4 \sin^3 x
\end{aligned}$$

5.

$$\mathbf{\cos 3x = 4 \cos^3 x - 3 \cos x}$$

We have,

$$\begin{aligned}
\cos 3x &= \cos (2x + x) \\
&= \cos 2x \cos x - \sin 2x \sin x \\
&= (2\cos^2 x - 1) \cos x - 2\sin x \cos x \sin x \\
&= (2\cos^2 x - 1) \cos x - 2\cos x (1 - \cos^2 x) \\
&= 2\cos^3 x - \cos x - 2\cos x + 2 \cos^3 x \\
&= 4\cos^3 x - 3\cos x.
\end{aligned}$$

6.

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} \text{ if } 3x \neq n\pi + \frac{\pi}{2}, \text{ where n is an integer}$$

We have $\tan 3x = \tan(2x + x)$

$$= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} = \frac{\frac{2\tan x}{1-\tan^2 x} + \tan x}{1 - \frac{2\tan x \cdot \tan x}{1-\tan^2 x}}$$

$$= \frac{2\tan x + \tan x - \tan^3 x}{1 - \tan^2 x - 2\tan^2 x} = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

Example 1: Prove that : $\dfrac{1+\sin 2\theta + \cos 2\theta}{1+\sin 2\theta - \cos 2\theta} = \cot\theta$

Solution: LHS= $\dfrac{1+\sin 2\theta + \cos 2\theta}{1+\sin 2\theta - \cos 2\theta} = \dfrac{1+\cos 2\theta + \sin 2\theta}{1-\cos 2\theta + \sin 2\theta}$

$$= \frac{2\cos^2\theta + 2\sin\theta\cos\theta}{2\sin^2\theta + 2\sin\theta\cos\theta} = \frac{2\cos\theta(\cos\theta + \sin\theta)}{2\sin\theta(\cos\theta+\sin\theta)}$$

$= \cot\theta$ =RHS.

Example2: Prove that: $\dfrac{\cos x}{1+\sin x}=\tan\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)$.

Solution: LHS= $\dfrac{\cos x}{1+\sin x}=\dfrac{\sin\left(\dfrac{\pi}{2}-x\right)}{1+\cos\left(\dfrac{\pi}{2}-x\right)}$

$$=\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}$$

$$=\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$$

= RHS

Example 3: Show that: $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}}=2\cos\theta$.

Solution: LHS= $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}}$

$$=\sqrt{2+\sqrt{2+\sqrt{2(1+\cos 8\theta)}}}$$

$$=\sqrt{2+\sqrt{2+\sqrt{4\cos^2 4\theta}}}$$

$$=\sqrt{2+\sqrt{2+2\cos 4\theta}}$$

$$=\sqrt{2+\sqrt{4\cos^2 2\theta}}$$

$$=\sqrt{2+2\cos 2\theta}$$

$$=2\cos\theta$$

=RHS

Example 4: Prove that: $(\cos A+\cos B)^2+(\sin A-\sin B)^2=4\cos^2\left(\frac{A+B}{2}\right)$

Solution: We have,

LHS = $(\cos A+\cos B)^2+(\sin A-\sin B)^2$

$=(\cos^2 A+\cos^2 B+2\cos A\cos B)+(\sin^2 A+\sin^2 B-2\sin A\sin B)$

$=(\cos^2 A+\sin^2 A)+(\cos^2 B+\sin^2 B)+2(\cos A\cos B-\sin A\sin B)$

$=2+2\cos(A+B)$

$=2[1+\cos(A+B)]$

$=4\cos^2(\frac{A+B}{2})$

3. Summary

In this module some formulae of trigonometric functions of multiple and submultiple angles have been derived. Using these identities, some results have been proved.