1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics (Class XI, Semester - 1)
Module Name/Title	Trigonometry: Part 4
Module Id	kemh_10304
Pre-requisites	Knowledge about Trigonometric Functions.
Objectives	After going through this lesson, the learners will be able to understand the following: 1.Introduction 2. Sine and Cosine Functions of Sum and Difference of Two Angles 3.Tangent and Cotangent Functions of Sum and Difference of Two Angles 4.Summary
Keywords	Multiple of an angle, Submultiple of an angle

2. Development Team

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TABLE OF CONTENTS

1. Introduction

In this module we will derive the identities based on the sum and difference of angles of trigonometric functions and use them in deriving some identities based on them. The first step will be to prove the identity $\cos(x + y) = \cos x \cos y - \sin x \sin y$. Using this, other formulae relating sum and difference of two angles will be proved.

2. Sine and Cosine Functions of Sum and Difference of Two Angles

In this section, we shall derive expressions for trigonometric functions of the sum and difference of two numbers (angles) and related expressions. The basic results in this connection are called trigonometric identities. We have seen that

1. **$\sin(-x) = -\sin x$**

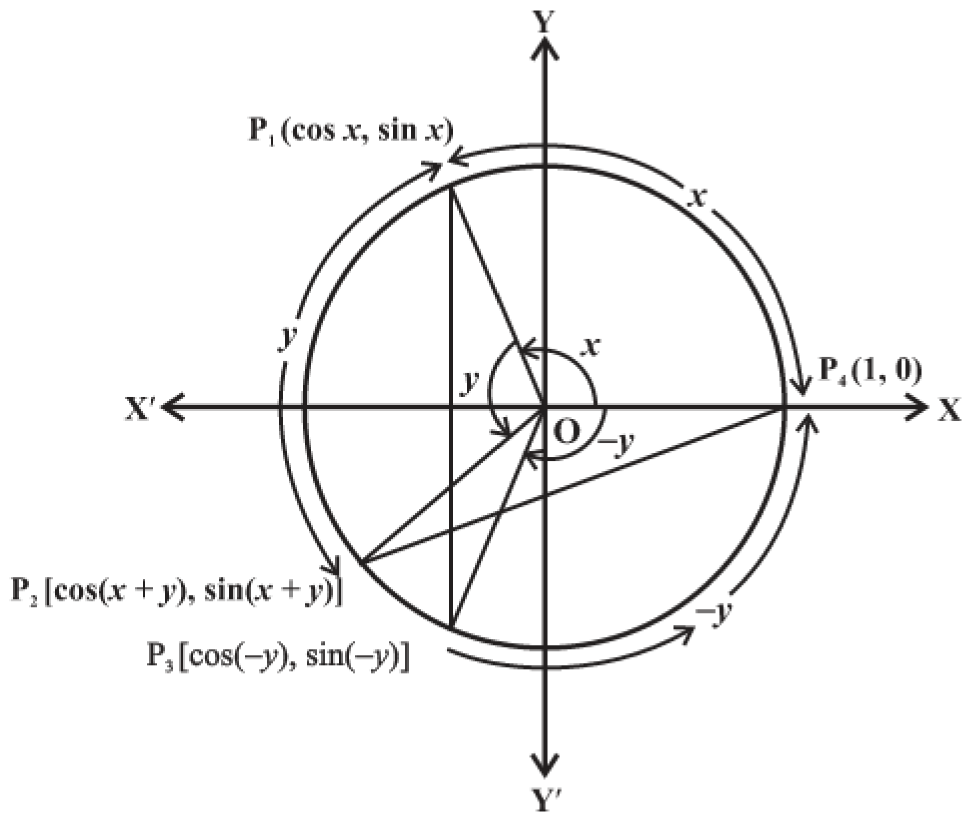
2. **$\cos(-x) = \cos x$**

We shall now prove some more results:

3. **$\cos(x + y) = \cos x \cos y - \sin x \sin y$**

$\cos(x + y) = \cos x \cos y - \sin x \sin y$

Consider the unit circle with centre at the origin



Let x be the angle P_4OP_1 and y be the angle P_1OP_2. Then (x + y) is the angle P_4OP_2. Also let (– y) be the angle P_4OP_3 . Therefore, P_1 , P_2 and P_3 and P_4 will have the coordinates P_1 (cos x, sin x), P_2 [cos (x + y), sin (x + y)], P_3 [cos (– y), sin (– y)] and P_4 (1, 0) as shown in the above figure.

Consider the triangles P_1OP_3 and P_2OP_4. They are congruent (by SAS). Therefore, P_1 P_3 and P_2 P_4 are equal. By using distance formula, we get

$$P_1P_3^2 = [\cos x - \cos(-y)]^2 + [\sin x - \sin(-y)]^2$$

$$= (\cos x - \cos y)^2 + (\sin x + \sin y)^2$$
$$= \cos^2 x + \cos^2 y - 2\cos x \cos y + \sin^2 x + \sin^2 y + 2\sin x \sin y$$
$$= 2 - 2(\cos x \cos y - \sin x \sin y)$$

Also,

$$P_2P_4^2 = [1 - \cos(x + y]^2 + [0 - \sin(x + y]^2$$
$$= 1 + \cos^2(x + y) - 2\cos(x + y) + \sin^2(x + y)$$
$$= 2 - 2\cos(x + y)$$

Since, $P_1 P_3 = P_2 P_4$ and also $P_1P_3^2 = P_2P_4^2$ (Proved above)
Therefore, 2 –2 (cos *x* cos *y* – sin *x* sin *y*) = 2 – 2 cos (*x* + *y*).
Hence, cos (*x* + *y*) = cos *x* cos *y* – sin *x* sin *y* ………..(3)

4. cos (*x* – *y*) = cos *x* cos *y* + sin *x* sin *y*
cos (*x* – *y*) = cos *x* cos *y* + sin *x* sin *y*

Replacing *y* by – *y* in identity 3, we get
cos (*x* + (– *y*)) = cos *x* cos (– *y*) – sin *x* sin (– *y*)
or, cos (*x* – *y*) = cos *x* cos *y* + sin *x* sin *y*… … .....(4)

5. $\cos\left(\frac{\pi}{2}-x\right)=\sin x$

If we replace *x* by $\frac{\pi}{2}$ and *y* by *x* in Identity (4), we get

$$\cos\left(\frac{\pi}{2}-x\right)=\cos\frac{\pi}{2}\cos x+\sin\frac{\pi}{2}\sin x=\sin x\ \ldots\ldots\ldots(5)$$

6. $\sin\left(\frac{\pi}{2}-x\right)=\cos x$

Using the Identity 5, we have

$$\sin\left(\frac{\pi}{2}-x\right)=\cos[\frac{\pi}{2}-\left(\frac{\pi}{2}-x\right)]=\cos x\ \ldots\ldots..(6)$$

7. sin (*x* + *y*) = sin *x* cos *y* + cos *x* sin *y*

$$\sin(x+y)=\cos\left(\frac{\pi}{2}-(x+y)\right)=\cos\left((\frac{\pi}{2}-x)-y\right)$$
$$=\cos(\frac{\pi}{2}-x)\cos y+\sin(\frac{\pi}{2}-x)\sin y$$
$$=\sin x\cos y+\cos x\sin y \qquad \ldots\ldots..(7)$$

8. sin (*x* – *y*) = sin *x* cos *y* – cos *x* sin *y*

If we replace *y* by –*y*, in the Identity 7, we get the result.

9. By taking suitable values of *x* and *y* in the identities 3, 4, 7 and 8, we get the following results:

$$\cos\left(\frac{\pi}{2}+x\right)=-\sin x \qquad \sin\left(\frac{\pi}{2}+x\right)=\cos x$$

$$\cos(\pi-x)=-\cos x \qquad \sin(\pi-x)=\sin x$$

$\cos(\pi + x) = -\cos x \qquad \sin(\pi + x) = -\sin x$

$\cos(2\pi - x) = \cos x \qquad \sin(2\pi - x) = -\sin x$

Example 1: If $\sin A = \frac{3}{5}, 0 < A < \frac{\pi}{2}$ *and* $\cos B = -\frac{12}{13}, \pi < B < \frac{3\pi}{2}$, find sin(A-B).

Solution:

$\sin A = \frac{3}{5}, \quad 0 < A < \frac{\pi}{2} \qquad \Rightarrow \cos A = \frac{4}{5}$

$\cos B = -\frac{12}{13}, \pi < B < \frac{3\pi}{2} \Rightarrow \sin B = -\frac{5}{13}$

$$\begin{aligned}\sin(A - B) &= \sin A \cos B - \cos A \sin B \\ &= \left(\frac{3}{5} \times -\frac{12}{13}\right) - \left(\frac{4}{5} \times -\frac{5}{13}\right) \\ &= -\frac{16}{65}\end{aligned}$$

Example 2: If $\sin A = \frac{3}{5}, 0 < A < \frac{\pi}{2}$ *and* $\cos B = -\frac{12}{13}, \pi < B < \frac{3\pi}{2}$,find cos(A+B).

Solution:

$\sin A = \frac{3}{5}, 0 < A < \frac{\pi}{2} \Rightarrow \cos A = \frac{4}{5}$

$\cos B = -\frac{12}{13}, \pi < B < \frac{3\pi}{2} \Rightarrow \sin B = -\frac{5}{13}$

$$\begin{aligned}\cos(A + B) &= \cos A \cos B - \sin A \sin B \\ &= \left(\frac{4}{5} \times -\frac{12}{13}\right) - \left(\frac{3}{5} \times -\frac{5}{13}\right) \\ &= -\frac{33}{65}\end{aligned}$$

Example 3: Find $\sin 75^0$ and $\cos 75^0$.

Solution: $\sin 75^0 = \sin(45^0+30^0) = \sin 45^0 \cos 30^0 + \cos 45^0 \sin 30^0$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$\cos 75^0 = \cos(45^0+30^0) = \cos 45^0 \cos 30^0 - \sin 45^0 \sin 30^0$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Example 4: Evaluate: $\cos\frac{2\pi}{3}\cos\frac{\pi}{4}-\sin\frac{2\pi}{3}\sin\frac{\pi}{4}$

Solution:

$$\cos\frac{2\pi}{3}\cos\frac{\pi}{4}-\sin\frac{2\pi}{3}\sin\frac{\pi}{4}$$
$$=\cos(\frac{2\pi}{3}+\frac{\pi}{4})=\cos\frac{11\pi}{12}$$
$$=\cos 165^0=-\cos 15^0$$

Now

$$\cos 15^0=\cos(45^0-30^0)$$

$$=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\therefore \cos\frac{2\pi}{3}\cos\frac{\pi}{4}-\sin\frac{2\pi}{3}\sin\frac{\pi}{4}=-\frac{\sqrt{3}+1}{2\sqrt{2}}$$

3. Tangent and Cotangent Functions of Sum and Difference of Two Angles

Similar results for tan *x*, cot *x*, sec *x* and cosec *x* can be obtained from the results of sin *x* and cos *x*.

10. If none of the angles *x*, *y* and (*x* + *y*) is an odd multiple of $\frac{\pi}{2}$, then

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

Since none of the *x*, *y* and (*x* + *y*) is an odd multiple of $\frac{\pi}{2}$, it follows that cos *x*, cos *y* and cos (*x* + *y*) are non-zero. Now

$$\tan(x+y)=\frac{\sin(x+y)}{\cos(x+y)}=\frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y}$$

Dividing numerator and denominator by cos *x* cos *y*, we have

$$\tan(x+y)=\frac{\frac{\sin x\cos y}{\cos x\cos y}+\frac{\cos x\sin y}{\cos x\cos y}}{\frac{\cos x\cos y}{\cos x\cos y}-\frac{\sin x\sin y}{\cos x\cos y}}$$
$$=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

………(10)

11. $\tan(x-y)=\dfrac{\tan x-\tan y}{1+\tan x\tan y}$

If we replace y by $-y$ in Identity 10, we get $\tan(x-y)=\tan[x+(-y)]=\dfrac{\tan x-\tan y}{1+\tan x\tan y}$

12. If none of the angles x, y and $(x+y)$ is a multiple of π, then

$$\cot(x+y)=\frac{\cot y\cot x-1}{\cot y-\cot x}$$

Since,

If none of the x, y and $(x+y)$ is multiple of π, we find that $\sin x \sin y$ and $\sin(x+y)$ are non-zero. Now

$$\cot(x+y)=\frac{\cos(x+y)}{\sin(x+y)}=\frac{\cos x\cos y-\sin x\sin y}{\sin x\cos y+\cos x\sin y}$$

Dividing numerator and denominator by $\sin x \sin y$, we have

$$\cot(x+y)=\frac{\cot y\cot x-1}{\cot y+\cot x} \quad \text{........(12)}$$

13. $\cot(x-y)=\dfrac{\cot x\cot y-1}{\cot y+\cot x}$ if none of angles x, y and $x-y$ is a multiple of π

If we replace y by $-y$ in identity 12, we get the result

$$\cot(x-y)=\frac{\cot y\cot x+1}{\cot y-\cot x}$$

Example5: Evaluate: $\tan 15^0$

Solution:

$$\tan 15^0=\tan\left(45^0-30^0\right)$$

$$=\frac{\tan 45^0-\tan 30^0}{1+\tan 45^0\tan 30^0}$$

$$=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}$$

$$=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

Example 6: Show that tan3x tan2x tanx = tan3x – tan2x –tanx.

Solution: We know that

3x = 2x + x

Or $\tan 3x = \dfrac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$

Or tan3x- tan3x tan2x tanx = tan2x - tanx

Or tan3x- tan2x – tanx = tan3x tan2x tanx

Or tan3x tan2x tanx = tan3x- tan2x – tanx

Example7: If $A + B = \dfrac{\pi}{4}$, prove that $(1+\tan A)(1+\tan B) = 2$.

Solution: We have

$$A + B = \frac{\pi}{4}$$

$$\Rightarrow \tan(A+B) = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$$

$$\Rightarrow \tan A + \tan B = 1 - \tan A \tan B$$

$$\Rightarrow \tan A + \tan B + \tan A \tan B = 1$$

$$\Rightarrow 1 + \tan A + \tan B + \tan A \tan B = 2$$

$$\Rightarrow (1+\tan A)(1+\tan B) = 2$$

4. Summary:

In this module identities based on the trigonometric functions of sum and difference of two angles have been discussed. These identities have been used to solve various questions.