1. **Details of Module and its structure**

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics (Class XI, Semester - 1)
Module Name/Title	Trigonometry: Part 2
Module Id	kemh_10302
Pre-requisites	Knowledge about Trigonometric Functions.
Objectives	After going through this lesson, the learners will be able to understand the following: 1. Trigonometric functions 2. Signs of Trigonometric functions 3. Summary
Keywords	Multiple of an angle, Submultiple of an angle.

2. **Development Team**

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1. Introduction

In this module we will see how the signs of different trigonometric functions change as we move from one quadrant to another. We will discuss the domain and range of trigonometric functions. Using domain and range, we will plot their graphs and check their periodicity.

2. Trigonometric functions

Consider a unit circle with center at origin of the coordinate axes.

Let P (a, b) be any point on the circle with angle AOP = x radian, i.e., length of arc AP = x (Figure 1). We define $\cos x$ = a and $\sin x$ = b. Since ΔOMP is a right triangle, we have $OM^2 + MP^2 = OP^2$ or $a^2 + b^2 = 1$. Thus, for every point on the unit circle, we have

$$a^2 + b^2 = 1 \text{ or } \cos^2 x + \sin^2 x = 1$$

Since one complete revolution subtends an angle of 2π radian at the centre of the circle, ∠AOB = $\frac{\pi}{2}$, ∠AOC = π and ∠AOD = $\frac{3\pi}{2}$.

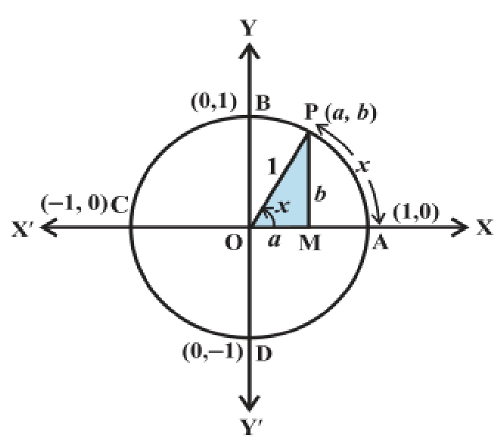

Figure 1

All angles which are integral multiples of $\frac{\pi}{2}$ are called quadrantal angles.

The coordinates of the points A, B, C and D are, respectively, (1, 0), (0, 1), (–1, 0) and (0, –1). Therefore, for quadrantal angles, we have

$\cos 0^\circ = 1$, $\sin 0^\circ = 0$ $\cos \frac{\pi}{2} = 0$, $\sin \frac{\pi}{2} = 1$, $\cos \frac{3\pi}{2} = 0$, $\sin \frac{3\pi}{2} = -1$, $\cos \pi = -1$, $\sin \pi = 0$,

$\cos 2\pi = 1$, $\sin 2\pi = 0$.

Now, if we take one complete revolution from the point P, we again come back to same point P. Thus, we also observe that if x increases (or decreases) by any integral multiple of 2π, the values of sine and cosine functions do not change. Thus,

$$\sin (2n\pi + x) = \sin x,\ n \in Z,\ \cos (2n\pi + x) = \cos x,\ n \in Z$$

Further, $\sin x = 0$, if $x = 0, \pm \pi, \pm 2\pi, \pm 3\pi, ...$, i.e., when x is an integral multiple of π and

$\cos x = 0$, if $x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \pi, ...$ i.e., $\cos x$ vanishes when x is an odd multiple of $\frac{\pi}{2}$. Thus

$\sin x = 0$ implies $x = n\pi$, where n is any integer and $\cos x = 0$ implies $x = (2n + 1)\frac{\pi}{2}$, where n is any integer.

We now define other trigonometric functions in terms of sine and cosine functions:

$\text{cosec}\, x = \frac{1}{\sin x}$, $x \neq n\pi$, where n is any integer.

$\sec x = \frac{1}{\cos x}$, $x \neq (2n+1)\frac{\pi}{2}$, where n is any integer.

$\cot x = \frac{\cos x}{\sin x}$ $x \neq (2n+1)\frac{\pi}{2}$ where n is any integer.

$\tan x = \frac{\sin x}{\cos x}$, $x \neq n\pi$, where n is any integer.

Example 1: Find the value of $\sin\frac{19\pi}{3}$.

Solution: $\sin\frac{19\pi}{3} = \sin\left(6\pi + \frac{\pi}{3}\right) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$.

Example 2: Find the value of cos 1470^0.

Solution: $1470^\circ = 4\times 360^\circ + 30^\circ$

Therefore, $\cos\left(1470^\circ\right) = \cos(4\times 360^\circ + 30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$.

Example3: Prove that $\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{3} - \tan^2\frac{\pi}{4} = -\frac{1}{2}$.

Solution: We have,

$$\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{3} - \tan^2\frac{\pi}{4}$$

$$= (\sin\frac{\pi}{6})^2 + (\cos\frac{\pi}{3})^2 - (\tan\frac{\pi}{4})^2$$

$$= (\frac{1}{2})^2 + (\frac{1}{2})^2 - (1)^2 = \frac{1}{4} + \frac{1}{4} - 1$$

$$= \frac{1}{2} - 1 = -\frac{1}{2}.$$

3. Sign of trigonometric functions

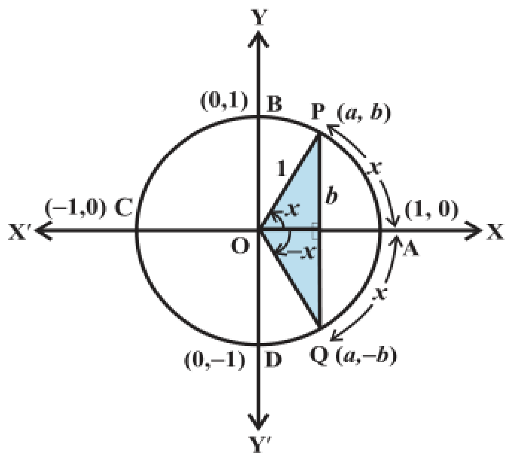

Figure 2

{To see in how sign changes as P moves from one quadrant to another: https://www.geogebra.org/m/mqvW2mPw}

Let P (a, b) be a point on the unit circle with centre at the origin such that $\angle AOP = x$. If $\angle AOQ = -x$, then the coordinates of the point Q will be (a, –b) (Figure 2). Therefore $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$. Since for every point P (a, b) on the unit circle, $-1 \le a \le 1$ and $-1 \le b \le 1$, we have $-1 \le \cos x \le 1$ and $-1 \le \sin x \le 1$ for all x. We have learnt in previous classes that in the first quadrant ($0 < x < \frac{\pi}{2}$) a and b are both positive, in the second quadrant ($\frac{\pi}{2} < x < \pi$) a is negative and b is positive , in the third quadrant ($\pi < x < \frac{3\pi}{2}$) a and b are both negative and in the fourth quadrant ($\frac{3\pi}{2} < x < 2\pi$) a is positive and b is negative. Therefore, $\sin x$ is positive for $0 < x < \pi$ and negative for $\pi < x < 2\pi$ Similarly, $\cos x$ is positive for $0 < x < \frac{\pi}{2}$, negative for $\frac{\pi}{2} < x < \frac{3\pi}{2}$ and also positive for $\frac{3\pi}{2} < x < 2\pi$.

Likewise, we can find the signs of other trigonometric functions in different quadrants. They are as follows:

Trigonometric Function	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
$\sin x$	+	+	–	–
$\cos x$	+	–	–	+
$\tan x$	+	–	+	–
$\cot x$	+	–	+	–
$\sec x$	+	–	–	+
$\operatorname{cosec} x$	+	+	–	–

Further, we have following results, which can be derived. We state them without proof.

1. $\sin(\frac{\pi}{2}-\theta)=\cos\theta$

2. $\cos(\frac{\pi}{2}-\theta)=\sin\theta$

3. $\sin(\frac{\pi}{2}+\theta)=\cos\theta$

4. $\cos(\frac{\pi}{2}+\theta)=-\sin\theta$

5. $\sin(\pi-\theta)=\sin\theta$

6. $\cos(\pi-\theta)=-\cos\theta$

7. $\sin(\pi+\theta)=-\sin\theta$

8. $\cos(\pi+\theta)=-\cos\theta$

9. $\sin(-\theta)=-\sin\theta$

10. $\cos(-\theta)=\cos\theta$

We can easily derive similar results for other trigonometric functions.

Example 4 . Find the value of $\tan 480^\circ$

Solution: We have $\tan 480^\circ = \tan\left(360^\circ + 120^\circ\right) = \tan 120^\circ$

$$= \tan(90^\circ + 30^\circ) = -\cot 30^\circ = -\sqrt{3}.$$

Example 5: find the value of $\tan\frac{19\pi}{3}$

Solution: $\tan\frac{19\pi}{3} = \tan\left(6\pi + \frac{\pi}{3}\right) = \tan(\frac{\pi}{3}) = \sqrt{3}$.

Example 6: Find the value of $\cot 570^\circ$.

Solution: $\cot 570^\circ = \cot\left(360^\circ + 210^\circ\right) = \cot 210^\circ = \cot\left(180^\circ + 30^\circ\right)$

$$= \cot 30^\circ = \sqrt{3}.$$

Example 7: Find the value of $\sin(\frac{-11\pi}{3})$

Solution: $\sin\left(\frac{-11\pi}{3}\right) = -\sin\left(\frac{11\pi}{3}\right) = -\sin\left(4\pi - \frac{\pi}{3}\right) = -\sin(-\frac{\pi}{3}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$.

Example 8: Prove that $\frac{\cos(90^\circ + \theta)\sec\left(-\theta\right)\tan\left(180^\circ - \theta\right)}{\sec\left(360^\circ - \theta\right)\sin\left(180^\circ + \theta\right)\cot(90^\circ - \theta)} = -1$

Solution: we have LHS= $\frac{\cos(90^\circ + \theta)\sec\left(-\theta\right)\tan\left(180^\circ - \theta\right)}{\sec\left(360^\circ - \theta\right)\sin\left(180^\circ + \theta\right)\cot(90^\circ - \theta)}$

LHS= $\frac{\left(-\sin\theta\right)\sec\left(\theta\right)(-\tan\theta)}{\sec\left(\theta\right)(-\sin\theta)\tan(\theta)} = -1$=RHS

Example 9: Prove that $\cos\left(\frac{3\pi}{2} + x\right)\cos(2\pi + x)\{\cot\left(\frac{3\pi}{2} - x\right) + \cot(2\pi + x)\} = 1$

Solution: We have $\cos\left(\frac{3\pi}{2} + x\right)\cos(2\pi + x)\{\cot\left(\frac{3\pi}{2} - x\right) + \cot(2\pi + x)\}$

$$= \cos\left(\pi + \frac{\pi}{2} + x\right)\cos x\{\cot\left(\pi + \frac{\pi}{2} - x\right) + \cot x\}$$

$$= -\cos\left(\frac{\pi}{2} + x\right)\cos x\{\cot\left(\frac{\pi}{2} - x\right) + \cot x\}$$

$$= (\sin x)(\cos x)\{\tan x + \cot x\}$$

$$= \sin x \cos x\left[\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right]$$

$$= \sin x \cos x\left[\frac{\sin^2 x + \cos^2 x}{\cos x \sin x}\right]$$

$$= \sin x \cos x \times \frac{1}{\sin x \cos x} = 1$$

Example10: Prove that $\dfrac{\cos(\pi + x)\cos(-x)}{\sin(\pi - x)\cos\left(\frac{\pi}{2} + x\right)} = \cot^2 x$

Solution: We have LHS = $\dfrac{\cos(\pi + x)\cos(-x)}{\sin(\pi - x)\cos\left(\frac{\pi}{2} + x\right)}$

$$= \frac{(-\cos x) \times \cos(x)}{\sin x \times (-\sin x)} = \frac{-\cos^2 x}{-\sin^2 x} = \cot^2 x \quad RHS$$

4. Summary

- All angles which are integral multiples of $\frac{\pi}{2}$ are called quadrantal angles.
- Sin x = 0, if x = 0, ± π, ± 2π , ± 3π, ..., i.e., when x is an integral multiple of π. Thus, sin x = 0 implies x = nπ, where n is any integer i.e. sin x vanishes when x is an integral multiple of π.
- Cos x = 0, if $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\pi, \ldots$ i.e., cos x vanishes when x is an odd multiple of $\frac{\pi}{2}$.

  Thus cos x = 0 implies x = (2n + 1) $\frac{\pi}{2}$, where n is any integer i.e. cos x vanishes when x is an integral multiple of $\frac{\pi}{2}$.