1. Details of Module and its structure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 01 (Class XI, Semester - 1)
Module Name/Title	Relation and Functions - Part 4
Module Id	kemh_10204
Pre-requisite	Sets and its properties, Cartesian Products
Objectives	After going through this lesson, the learners will be able to solve: ● Problems based on Relation and Functions ● Find Domain and Range of given Relation and Function
Keywords	Cartesian Product

2. Development Team

Role	Name	Affiliation
National MOOC Coordinator (NMC)	Prof. Amarendra P. Behera	CIET, NCERT, New Delhi
Program Coordinator	Dr. Indu Kumar	CIET, NCERT, New Delhi
Course Coordinator (CC) / PI	Dr. Til Prasad Sarma	DESM, NCERT, New Delhi
Subject Coordinator	Anjali Khurana	CIET, NCERT, New Delhi
Subject Matter Expert (SME)	Dr. Monika Sharma	Shiv Nadar University, Noida
Revised By	Dr. Sadhna Srivastava (Retd.)	PGT, KVS, Faridabad
Review Team	Prof. Bhim Prakash Sarrah, Prof. Ram Avtar (Retd.) Prof. Mahendra Shankar (Redt.)	Assam University, Tezpur. DESM, NCERT, New Delhi DESM, NCERT, New Delhi

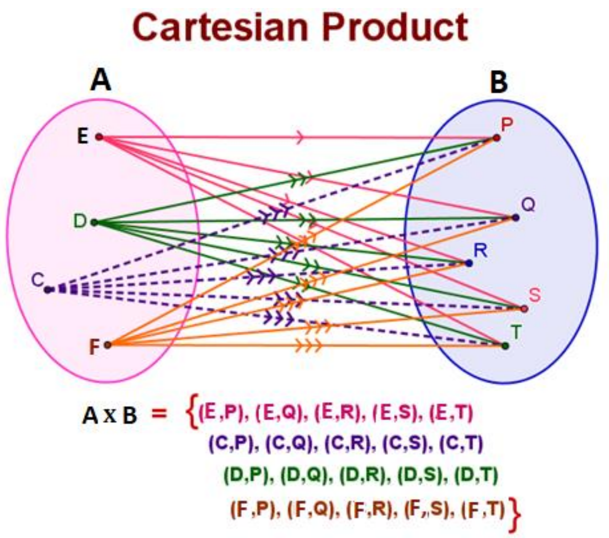
TABLE OF CONTENTS

1. Introduction:

In the previous modules, we have studied Cartesian product, Relations and functions. In this module we will solve some problems related with them and will learn to find out domain and range of a given function.

Before proceeding let us recapitulate what we have learnt in our previous modules.

The Cartesian product of two non-empty sets A and B is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ and is denoted by $A \times B$. We represent it diagrammatically by arrow diagram as below;

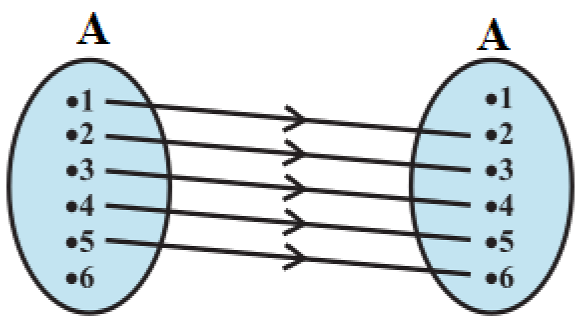


A relation R from a non-empty set A to a non-empty set B is a subset of Cartesian product A×B, which is derived by describing a relationship between the first element and the second element of the ordered pairs in A×B.

For example, if A = {1, 2, 3, 4, 5, 6}, then a relation R from set A to set A, defined by $R = \{(x, y): y = x + 1\}$ is;

$R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\} \subset A \times A$

and is depicted by the arrow diagram as;

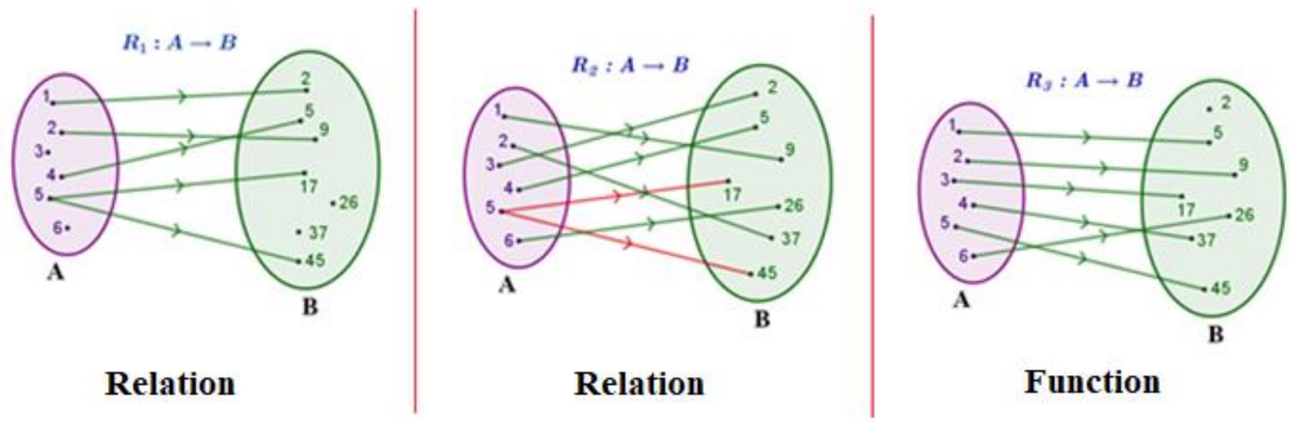


A relation from a set A to a set B is said to be a function if every element of set A has a unique image in set B. For example consider two sets A and B as;

A = {1, 2, 3, 4, 5, 6}

B = {2, 5, 9, 17, 26, 37, 45}

In the following diagram we have visual representation of three relations,



Comparison of visual representation of these three relations explains that relation R_3 is a special relation for which all the elements of set A have unique images in set B. Such special relations, as relation R_3 are known as functions.

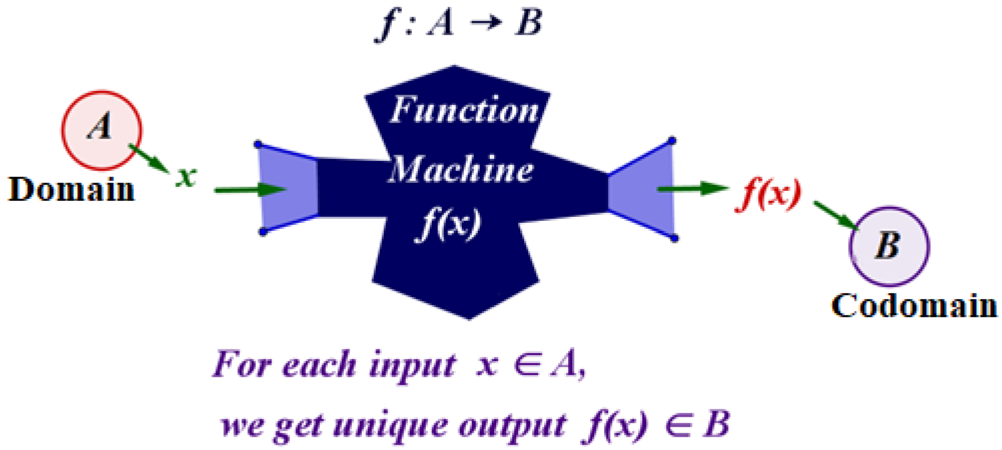
Recall, a function f from set A to set B is denoted by,

f: A → B, such that

$f = \{(x, f(x)): x \in A \text{ and } f(x) \in B\}$.

The set A is Domain, set B is Co-domain of function f and the set of all images $f(x)$ of $x \in A$ under f is the Range of the function f.

In our previous Modules, we have discussed that a function can be considered as a machine. For each input given from domain the function machine produces a unique output belonging to the codomain of the function as is shown below;



2. Problems Based on Relations and Functions

Example 1:

Find the range of the function;

$$f(x) = 2 - 3x, \qquad x \in R, x > 0$$

Solution:

The given function is;

$$f(x) = 2 - 3x, \qquad x \in R, x > 0$$

According to the question its domain is $(0, \infty)$.

To find its range let,

$$y = 2 - 3x, \qquad x \in R, x > 0$$

$\Rightarrow 3x = 2 - y$

$\Rightarrow 2 - y > 0 \qquad (\because x > 0, \therefore 3x > 0)$

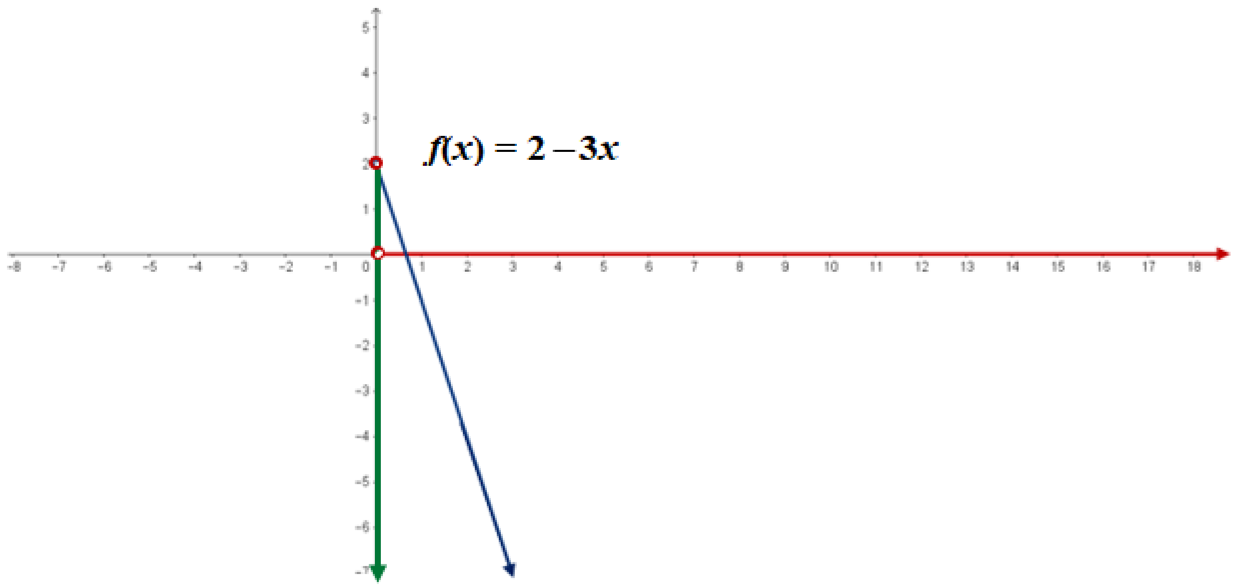
$\Rightarrow y < 2$

$\Rightarrow y = (-\infty, 2)$

Hence range of the given function denoted by R_f is;

$$R_f = (-\infty, 2)$$

Look the graph of the function given below, domain of the function $(0, \infty)$ is shown by red colour and range of the function $(-\infty, 2)$ is shown by green colour, $x = 0$ is shown by hollow circle because it is not included in domain. Similarly, $y = 2$ is also shown by hollow circle because it is not included in range of the function. The function is a linear function hence its graph is a straight line shown by blue colour in the picture.



Example 2:

Let $f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}$ be a linear function from Z into Z. Find $f(x)$.

Solution:

Since the function f is a linear function,

$\therefore$ let, $f(x) = mx + c$……………………(i)

Since, $(1, 1)$ and $(0, -1) \in$ R,

$\therefore f(1) = m + c = 1$ and $f(0) = c = -1$.

Solving we get, $m = 2$ and $c = -1$

Hence, from (i) we get, $f(x) = 2x - 1$.

(Observe, other two pairs also satisfy it.)

Example 3:

Let function f be defined by,

$$f(x) = 5x^2 + 2, x \in \mathrm{R},$$

Find the value of x for which $f(x) = 22$.

Also find the value of p if, $p = f(3) \times f(4)$

Solution:

$f(x) = 22 \Rightarrow 5x^2 + 2 = 22$

$\Rightarrow 5x^2 = 20$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

Hence, for $x = 2$ and -2, $f(x) = 22$.

Now, $f(3) = 5\times3^2 + 2 = 47$

And $f(4) = 5\times4^2 + 2 = 82$

Thus $p = f(3) \times f(4) = 47 \times 82 = 3854$

Example 4:

Let, $f = \{(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)\}$ be a function,

from set A to set B, where

A = {1, 2, 3, 4},

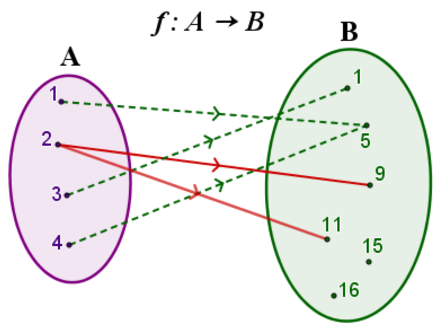
B = {1, 5, 9, 11, 15, 16}

Is f a function from A to B?

Solution:

We have, f: A → B, hence the set A is domain and set B is codomain.

For f to be a function from A to B, every element of set A must have a unique image in set B. But here two pairs, "(2, 9) and (2, 11)" belonging to the function have same first element, that means they do not have unique image in codomain, as is clear from the following visual representation.



Hence f is not a function, it is a relation from set A to set B.

Example 5:

Let A = {9, 10, 11, 12, 13} and f : A→N be defined by;

$f(n)$ = the highest prime factor of n. Find the range of f.

Solution:

Let us find the prime factors of the elements of set A.

$9 = 3\times3$, $10 = 2\times5$, $12 = 2\times3\times2$,

11 and 13 are prime numbers.

Thus, $f(n)$ = the highest prime factor of n gives,

$f(9) = 3$, $f(10) = 5$, $f(11) = 11$, $f(12) = 3$, $f(13) = 13$

hence, Range of f = {3, 5, 11, 13}

3. Problems Based on Finding Domain and Range for a Given Function

Example 6:

Find the domain for which the functions,

$f(x) = 2x^2 - 1$ and $g(x) = 1 - 3x$ are equal.

Solution:

Two functions f and g are called equal, if and only if,

(i) Domain of f = Domain of g

(ii) $f(x) = g(x)$ in the domain of f (or g)

For $f(x) = g(x)$

We have, $2x^2 - 1 = 1 - 3x$

$\Rightarrow 2x^2 + 3x - 2 = 0$

$\Rightarrow 2x^2 + 4x - x - 2 = 0$

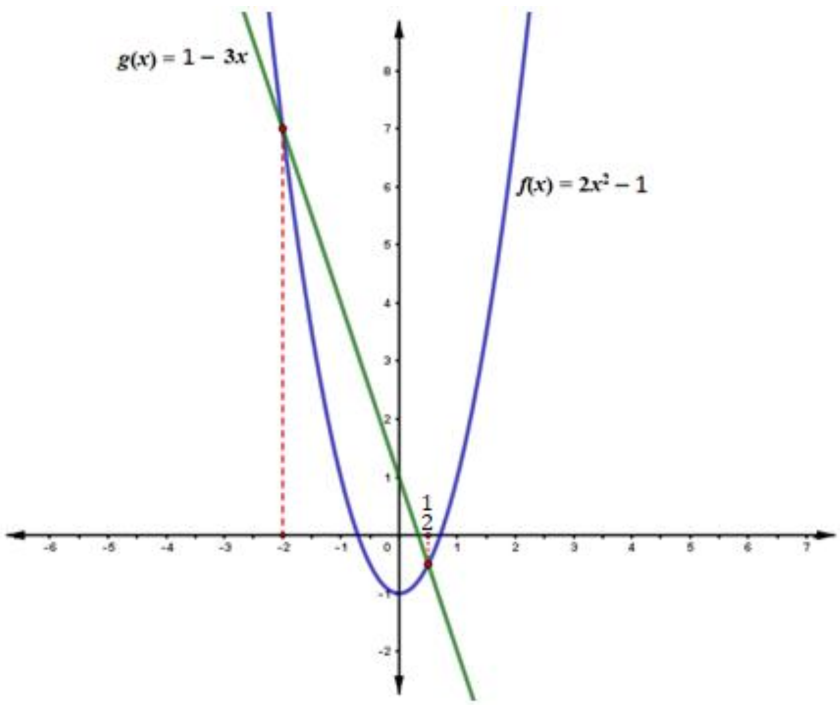
$\Rightarrow 2x(x + 2) - 1(x + 2) = 0$

$\Rightarrow (2x - 1)(x + 2) = 0$

Therefore, domain for which the function $f(x) = g(x)$, we must have

$(2x - 1) = 0$ and $(x + 2) = 0$

Or $x = \frac{1}{2}$ and $x = -2$



Thus domain for which given functions are equal is $\left\{\frac{1}{2}, -2\right\}$, which is obvious from the graph of the functions shown above.

Example 7:

Find Domain and range of the function;

$$f(x) = 3x^2 - 5,$$

Solution:

Domain of function 'f' must contain all those values of x for which

$f(x) = 3x^2 - 5$ is real.

We know, $3x^2 - 5$ is real for all $x \in \mathrm{R}$,

Hence, Domain of $f = \mathrm{R}$

To find range of the function, let

$y = f(x) = 3x^2 - 5$

$\Rightarrow 3x^2 = y + 5$

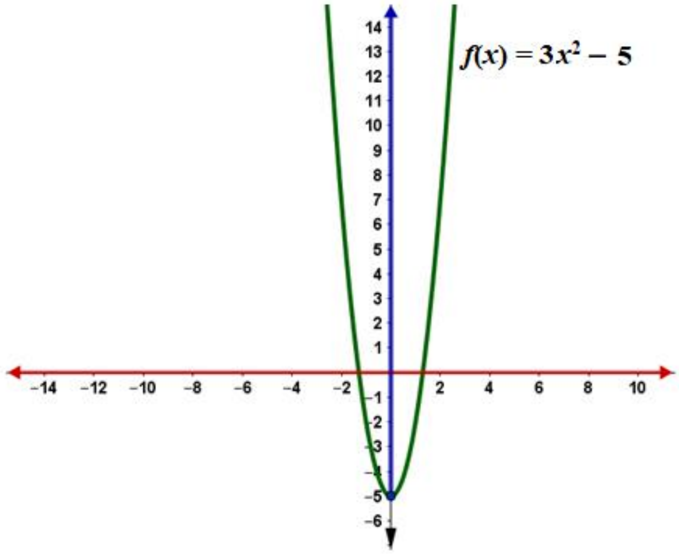
$\Rightarrow x = \sqrt{\frac{y+5}{3}}$

For $x \in \mathrm{R}$, we must have $y + 5 \geq 0$

$\Rightarrow y \geq -5$

Range of $f = [-5, \infty)$

In the graph below, observe the domain and range of the function. Domain of the function is shown by red colour and range of the function is shown by blue colour.



Example 8:

Find Domain and Range of the function;

$$f(x) = \sqrt{x-1},$$

Solution:

(i) The given function is,

$$f(x) = \sqrt{x-1}$$

For domain of function we have to find all those values of x for which $f(x)$ is real.

$\therefore \sqrt{x-1}$ must be real.

$\Rightarrow x - 1 \geq 0$

$\Rightarrow x \geq 1$

Hence, Domain of $f = [1, \infty)$

For range of the function, let

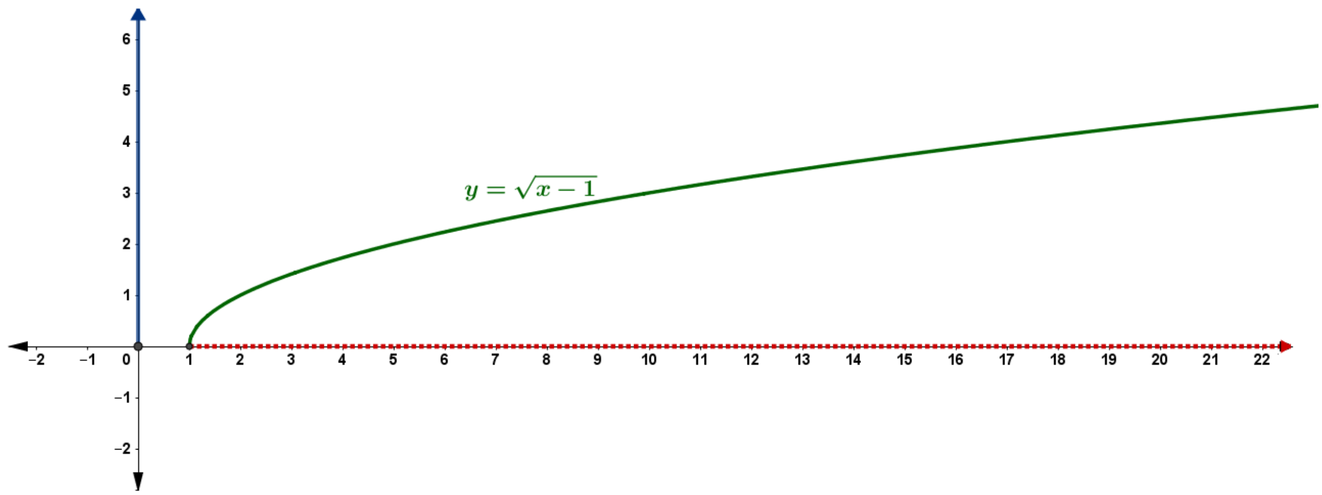
$$y = f(x) = \sqrt{x-1}$$

As $x \geq 1$, we get $x - 1 \geq 0$

$\Rightarrow y \geq 0$

Thus, Range of $f = [0, \infty)$

The graph of the function is shown below which shows domain of the function by red colour and range of the function by blue colour.



Example 9:

Find Domain and range of the function;

$$f(x) = \sqrt{16 - x^2},$$

Solution:

The given function is,

$f(x) = \sqrt{16 - x^2}$

For domain of function $f(x)$ must be real.

$\therefore \sqrt{16 - x^2}$ must be real.

$\Rightarrow 16 - x^2 \geq 0$

$\Rightarrow 16 \geq x^2$

$-4 \leq x \leq 4$

Hence, domain of the function = $[-4, 4]$

For range of the function, let $y = f(x) = \sqrt{16 - x^2}$

We know square root of a real number is always non-negative.

Therefore, $y \geq 0$

Squaring, $y = \sqrt{16 - x^2}$, we get,

$$y^2 = 16 - x^2$$

$\Rightarrow x^2 = 16 - y^2$

For all x belonging to domain of the function we have,

$$x^2 \geq 0$$

$\Rightarrow 16 - y^2 \geq 0$

$\Rightarrow y^2 - 16 \leq 0$

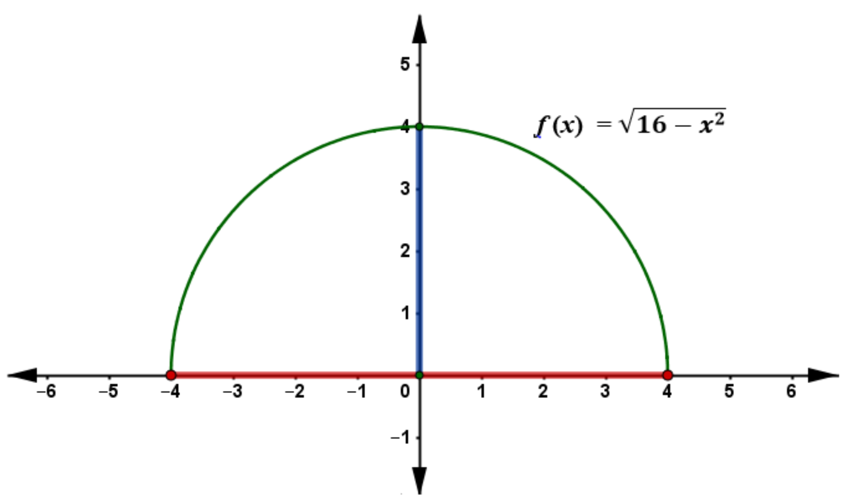
$\Rightarrow y^2 \leq 16$

$\Rightarrow -4 \leq y \leq 4$

But we have, $y \geq 0 \therefore 0 \leq y \leq 4$

Hence, Range of the function = [0, 4]

Look the graph of the function $f(x) = \sqrt{16 - x^2}$, shown below in green colour and see the domain is depicted by red colour and range is shown by blue colour.



Example 10:

Find Domain and range of the function;

$$f(x) = \frac{x-2}{3-x}$$

Solution:

The given function is,

$$f(x) = \frac{x-2}{3-x}$$

Clearly this function is defined for all $x \in$ R, except $x = 3$.

Thus, Domain of the function = R – {3}.

To find range of the function, let

$$y = f(x) = \frac{x-2}{3-x}$$

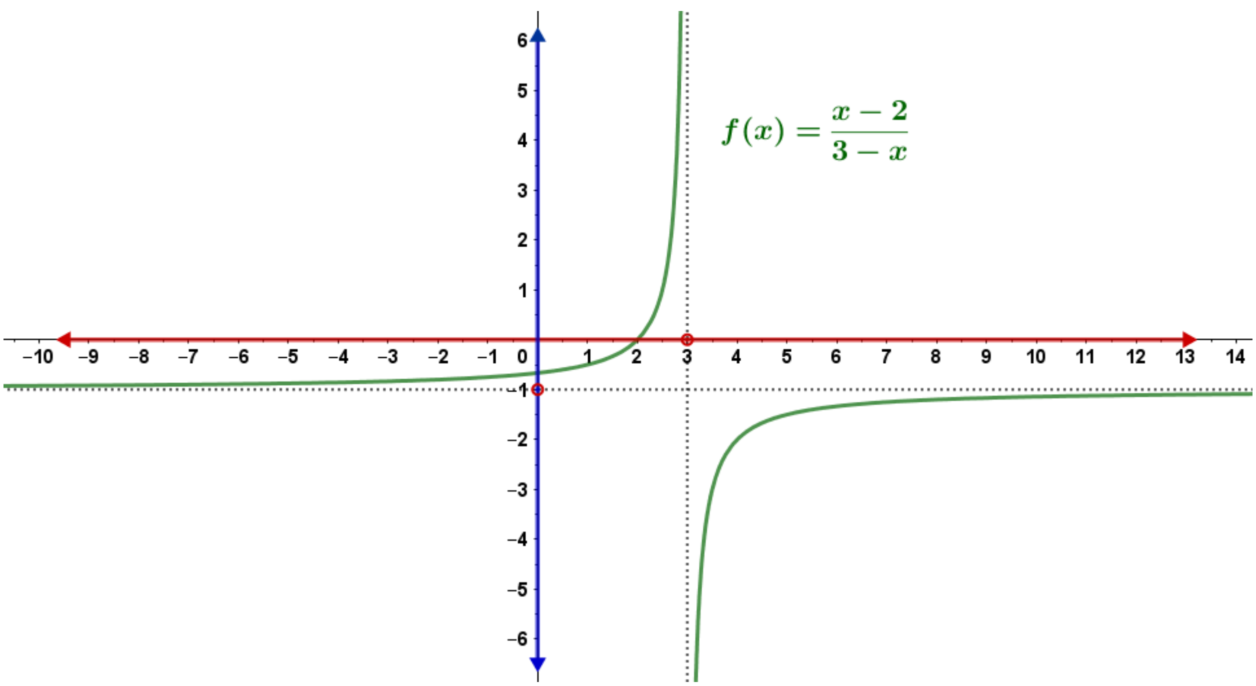
Solving we get,

$$x = \frac{3y+2}{y+1}$$

It is obvious from here that x will assume real values for all y

except, $y = -1$.

Hence, Range of the function = R – {– 1}.



Look at the graph of the function $f(x) = \frac{x-2}{3-x}$, the graph has two branches, shown by green colour, domain is shown by red colour and range by blue colour.

It is obvious from the graph that domain of the function is R except the real number "3", i.e. Domain = R – {3}.

And range of the function is R except the real number "– 1", i.e.

Range = R – {– 1}.

4. Summary:

1) The Cartesian product of two non-empty sets A and B is the set of all ordered pairs (a, b) such that $a \in$ A and $b \in$ B.

2) A relation R from a set A to a set B is a subset of the Cartesian product A × B obtained by describing a relationship between the first element x and the second element y of the ordered pairs in the set A × B.

3) A function f from a set A to a set B is a special type of relation for which every element x of set A has one and only one image y in set B.

4) A is the domain and B is the codomain of function f.

5) The range of the function is the set of images.

6) Domain of real functions are subsets of R.

7) Algorithm to find Range of a real function f ,

i) Put $y = f(x)$

ii) Solve the equation $y = f(x)$ for x in terms of y.

iii) Let $x = \varphi(y)$.

iv) Find values of y for which the values of x, obtained from $x = \varphi(y)$, are real and in the domain of f.

v) The set of values of y obtained in step (iv) is the range of f.