1. Details of Module and itsstructure

Module Detail	
Subject Name	Mathematics
Course Name	Mathematics 01 (Class XI, Semester - 1)
Module Name/Title	Relation and Functions - Part 3
Module Id	kemh_10203
Pre-requisite	Sets and its properties, Cartesian Products
Objectives	After going through this lesson, the learners will be able to understnd • Cartesian Product • Relation and Function
Keywords	Cartesian Product, Relation, Function, Domain, Co-domain and Range of function and Some types of functions

2. Development Team

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1. Introduction:

In our previous module we have learnt that a relation R defined from a set A to a set B is a subset of the Cartesian product A × B.

This subset is obtained by describing a relationship between the first element *x* and the second element *y* of the ordered pairs belonging to the Cartesian product A × B.

In this module, a special type of relation which is called a function is dicussed. It is one of the most important concepts in Mathematics.

The word function first appeared in a Latin manuscript written by G.W. Leibnitz (1646 - 1716) in 1673.

G. W. Leibnitz
(1646–1716)

The concept of function is very important in Mathematics since it captures the idea of a mathematically precise correspondence between one quantity with the other.

There are many terms such as ‘map’ or ‘mapping’ used to denote a function.

Since function is a special type of relation so before defining a function let us discuss some relations defined from set A to set B.

Consider two sets A and B,

A = {1, 2, 3, 4, 5, 6}

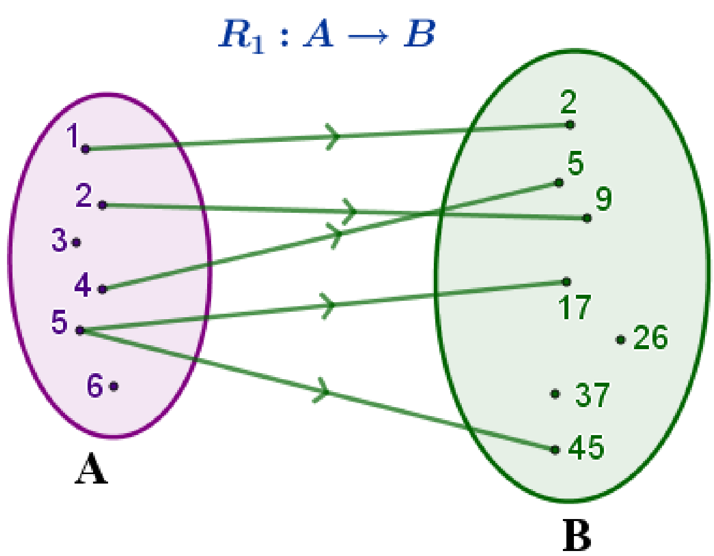
B = {2, 5, 9, 17, 26, 37, 45}

Since n(A) = 6 and n(B) = 7, therefore, with our previous knowledge we can say that there will be 6× 7 = 42, relations from set A to set B.

Let us define three relations R_1, R_2 and R_3 from set A to set B one by one,

R_1= {(1, 2),(2, 9), (4, 5), (5, 17),(5, 45)}

A visual representation of this relation is;



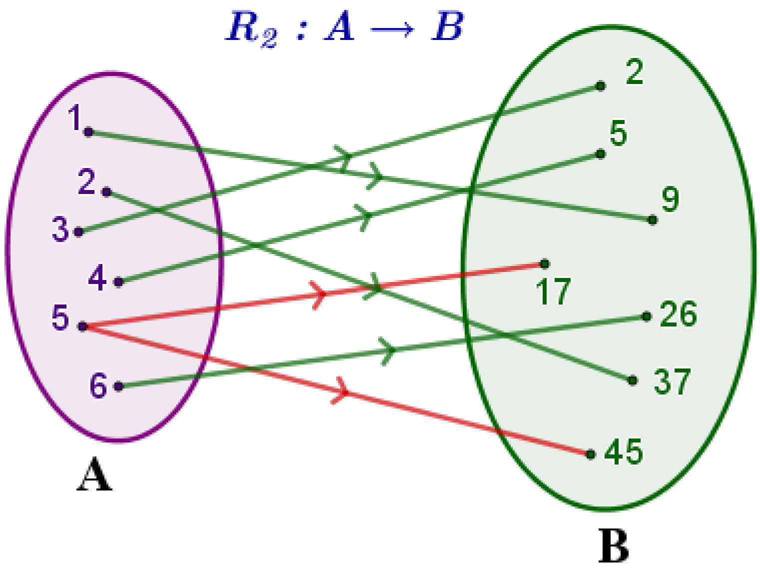
What do we observe?

Two elements of set A,‘3’ and ‘6’ are left without association with any of the elements of the set B under relation R_1 and element ‘5’ is associated with two elements of set B.

Now, let us consider another relation R_2, defined by

R_2= {(1,9),(2,37), (3, 2),(4,5), (5,17),(5, 45),(6,26)}

Visual representation of relation R_2 is;

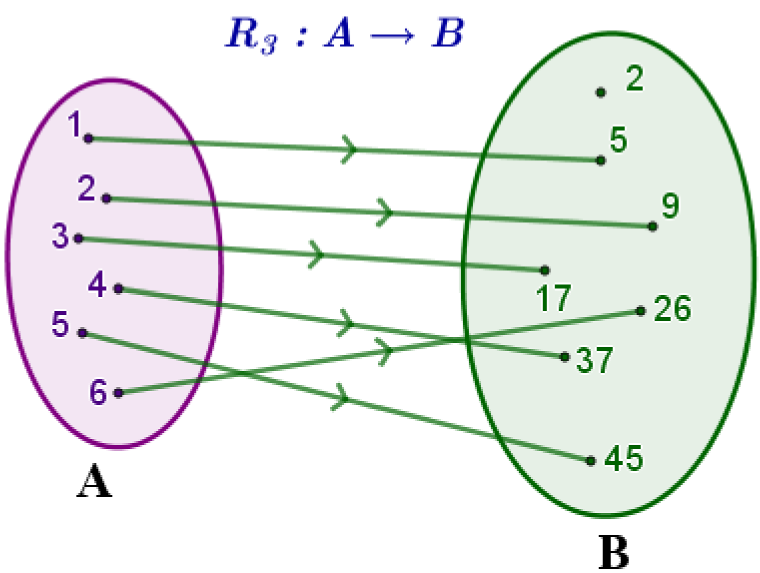


Under this relation all elements of set A have association with elements of set B but element ‘5’ is associated with two elements of set B.

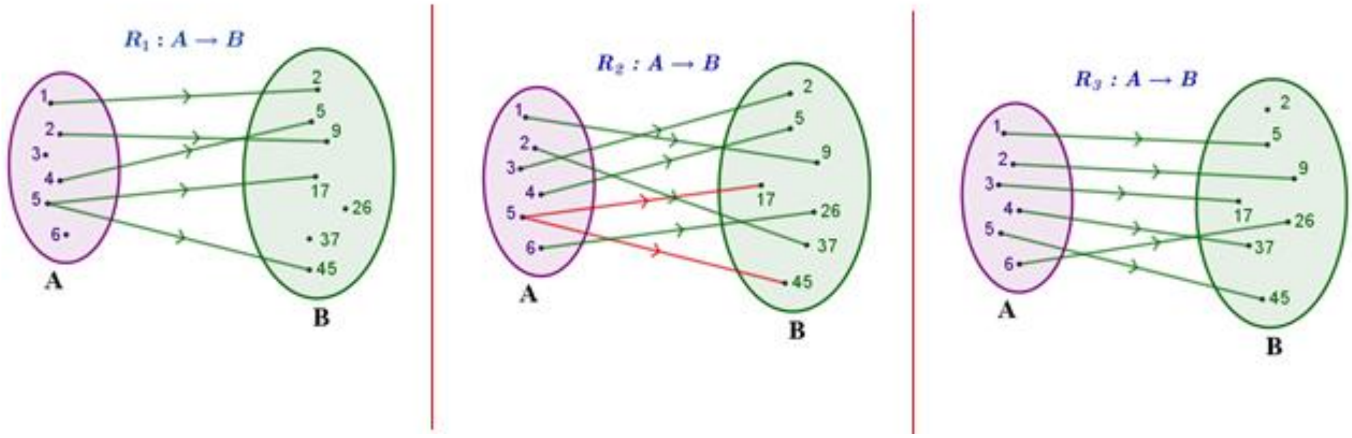
Now third relation R_3 is defined by,

R_3= {(1,5),(2,9), (3,17), (4,37), (5, 45),(6, 26)}

Look at visual representation of relation R_3, we find that all the elements of set A have association with elements of set B and no element of set A is associated with two elements of set B.



For better understanding let us compare the visual representation of these three relations is given below,



Comparison of visual representation of these three relations explains that relation R_3 is a special relation for which all the elements of set A have unique images in set B. No element of set A is left without association and no element of set A has association with two elements in set B like relations R_1 and R_2.

Such special relations, as relation R_3, are defined as functions.

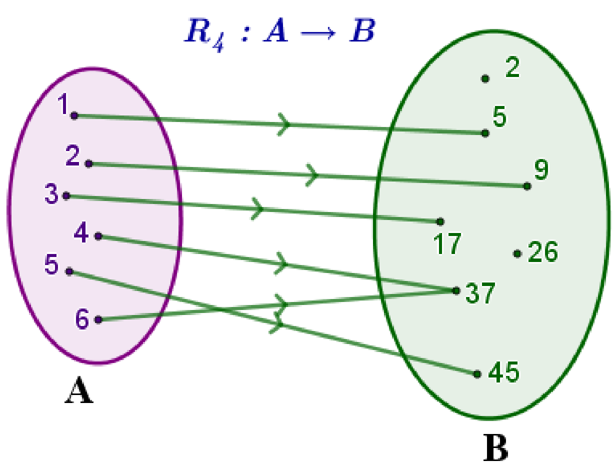
Function:

A relation from a set A to a set B is said to be a function if every element of set A has one and only association with the element in set B. In other words, every element of set A has one and only one image in set B.

Let us define one more relation R_4 from set A to set B,

$R_4 = \{(1, 5),(2, 9), (3, 17), (4, 37), (5, 45),(6,37)\}$

Observe the visual representation of relation R_4 carefully;



Though all the elements of set A have images in set B but two elements of set A ‘4’ and ‘6’ have same image in set B.

Can the relation R_4 be called a function?

Yes, relation R_4 represents a function because all the elements of set A have unique images in set B. In other words we can say that a relation for which no two distinct ordered pairs have the same first element defines a function.

A function f from set A to set B is denoted by,

f: A → B, such that

$f = \{(x, f(x)): x \in A \text{ and } f(x) \in B\}$.

Domain and Co-domain:

The set A is known as the Domain of function f and the set B is known as the Co-domain of function f.

Image and Pre-image:

If, f is a function from set A to set B and (a, b) $\in f$, where $a \in A, b \in B$

Then, $f(a) = b$ and b is called the image of a under function f and a is called the pre-image of b under f.

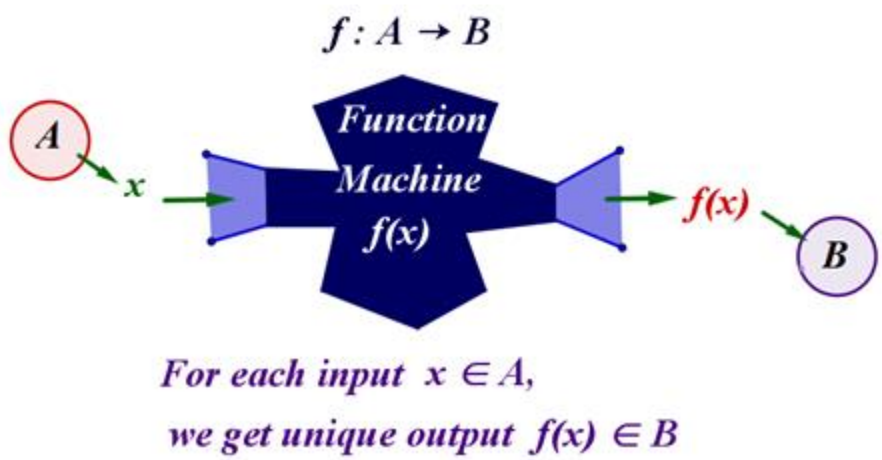
Range and Codomain:

The set of all images $f(x)$ of $x \in A$ under f from set A to set B is called the Range of the function f. The whole set B is the codomain of the function f. It is to note that Range ⊆ Codomain.

We can visualise a function as a rule, which produces new elements out of given elements.

Actually speaking function can be thought of a machine.

If we define function, f: A → B, then for each input $x \in A$ it will give a unique output $f(x) \in B$.



For each input given from domain the function machine produces unique output belonging to codomain.

Let us discuss some examples to understand the concept of function.

Example 1:

Let N be the set of natural numbers and the relation R be defined on N such that,

R = $\{(x, y): y = 2x, x, y \in N\}$.

What is the domain, codomain and range of R? Is this relation a function?

Solution:

Since R: N → N

∴ Domainof R =$\{x: x \in N\}$= Set of natural numbers N

Codomainof R =Set of natural numbers N

Relation R is defined on N by xRy if $y = 2x$, therefore,

Range of R = $\{y: y = 2x, x \in N\}$

= Set of even natural numbers

Since every natural number $n \in$Domain has one and only one image in Codomain, this relation is a function.

Example 2:

Following are some relations defined on set of real numbers 'R'.

Examine each of the following relations and state in each case whether it is a function or not?

(i) $R_1 = \{(2, 7),(5,1), (9,2)\}$,

(ii) $R_2 = \{(5,2),(2,4),(3, 7), (5,4)\}$

(iii) $R_3 = \{(1,2),(2,3),(3,4), (4,5), (5,6), (6,7)\}$

Give reasons to support your answer.

Solution:

(i) Since 2, 5, 9 are the elements of domain of R_1, having their unique images, so the relation R_1 is a function.

(ii) Since the same first element 5 corresponds to two different images 2 and 4. That means two distinct ordered pairs have the same first element, hence this relation is not a function.

(iii) Observing all the ordered pairs, we find that every element belonging to domain of relation R_3 has one and only one image, hence this relation is a function.

Real Function:

A function which has either R or one of its subsets as its range is called a real valued function. Further, if its domain is also either R or a subset of R, it is called a Real function.

Note:

1) Form the study of Cartesian product of two sets A and B,

The relation R: A → B and then the function *f*: A → B, we conclude that,

Function ⊆Relation ⊆Cartesian Product

2) Every function is a relation but every relation is not a function.

Example 3:

Let N be the set of natural numbers. A real valued function $f : N \rightarrow N$ is defined by $f(x) = 2x + 1$. Find the images for $x = 1, 5, 9$ and 24 under function *f*.

Solution:

Given that,function $f : N \rightarrow N$ is defined by $f(x) = 2x + 1$.

Hence, images for $x = 1, 5, 9$ and 24 under function *f*,

$f(1) = 3$, $f(5) = 11$, $f(9) = 19$, $f(24) = 49$.

Let us now learn some real functions along with their graphs.

Identity function:

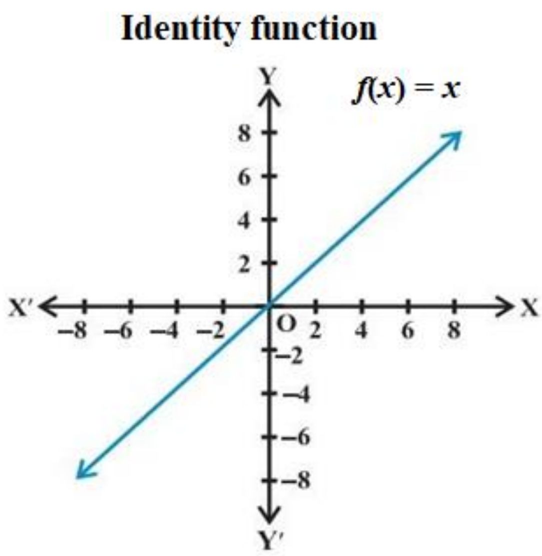
Let R be the set of real numbers. Define the real valued function,

f : R → R by $y = f(x) = x$ for each $x \in$ R.

Observe each element is associated to itself.

Such a function is called the identity function. Thedomain and range of f both are R.

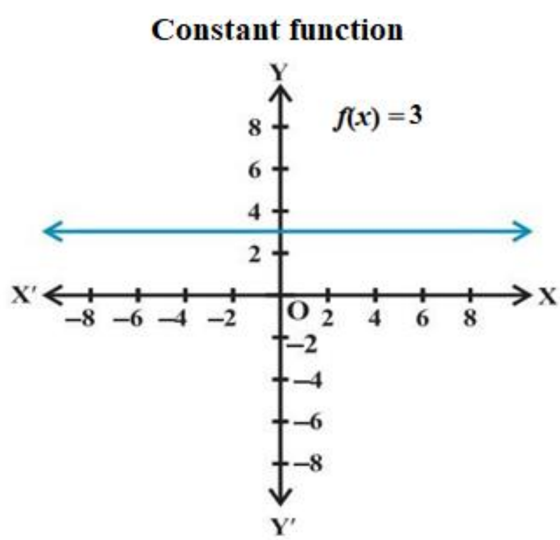
The graph of Identity function is a straight line which passes through the origin.



Constant function:

We define constant function f: R → R by $y = f(x) = c$, for each $x \in$ R where c is a constant.

Here domain of f is R and its range is$\{c\}$.The graph of this function will be a line parallel to x-axis.



For example, if $f(x)=3$ for each $x \in R$, then its graph will be a line parallel to x-axis at a distance $y = 3$ above x-axis.

Polynomial Function:

A function, $f: R \to R$ is said to be a polynomial function if for each

$x \in R$, $y = f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$, where n is a non-negative integer and $a_0, a_1, a_2, ..., a_n \in R$.

For example the functions defined by,

$f(x) = x^3 - x^2 + 2$ and $g(x) = x^4 + \sqrt{2}\,x$ are examples of polynomial functions, but the function defined by,

$h(x) = x^{\frac{2}{3}} + 2x$ is not a polynomial function as index of x is a fraction (not a non-negative integer).

Example 4:

Find domain and range of the function defined by,

$f: R \to R$ by $y = f(x) = x^2$, $x \in R$.

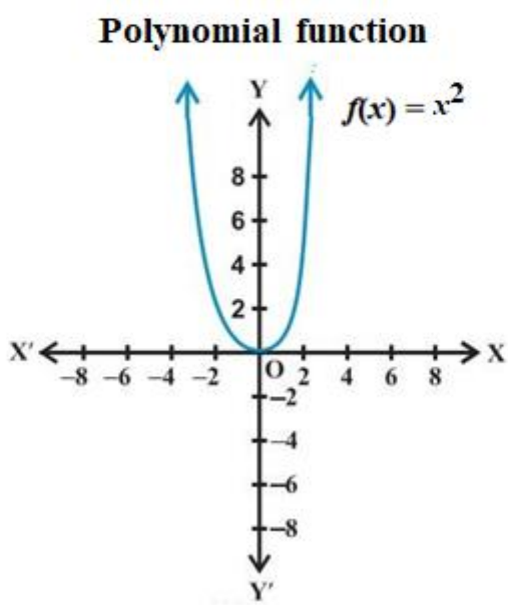
Also draw the graph of f.

Solution:

Domain of $f = \{x: x \in R\}$

Range of $f = \{y: y = x^2: x \in R\}$

The graph of f is given below:



Example 5:

Draw graph of the function defined by,

f: R→ R by $y = f(x) = x^3$, $x \in$ R.

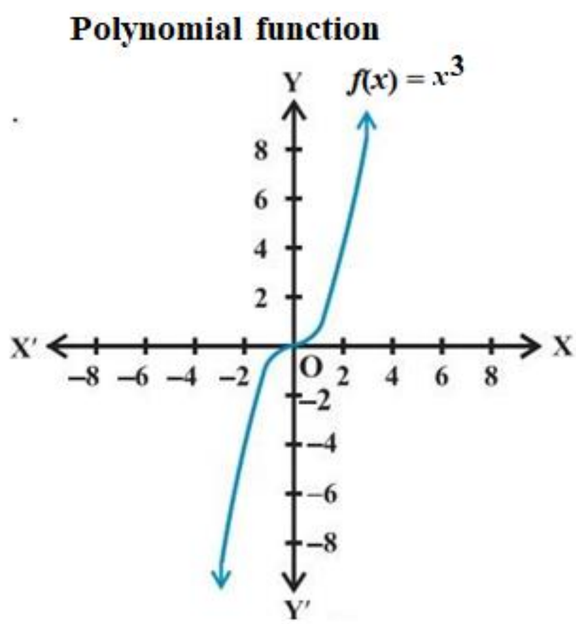
Solution:

Taking some different values for $x \in$ R, we get,

$f(0) = 0$, $f(1) = 1$, $f(-1) = -1$, $f(2) = 8$, $f(-2) = -8$,

$f(3) = 27$, $f(-3) = -27$, so on.

Therefore for $f = \{(x, x^3): x \in R\}$, the graph of f is given below,



Rational Functions:

Rational functions are functions of the type, $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomial functions of defined in a domain, where $g(x) \neq 0$.

Example 6:

A real valued function f: R –{0}→ R is defined by,

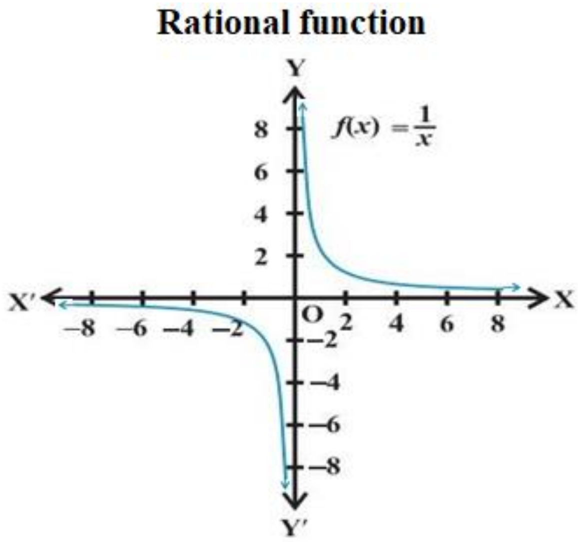
$y = f(x) = \frac{1}{x}$, $x \in$ R –{0}, Write domain and range of this function.

Also draw the graph for this function.

Solution: Domain of $f = \{x: x \in R - \{0\}\}$ = Set of all real numbers except 0.

Range of $f = \{y: y = \frac{1}{x}, x \in R - \{0\}\}$=Set of all real numbers except 0.

The graph of *f* is given below:



The Modulus Function:

The function *f*: R→R defined by,

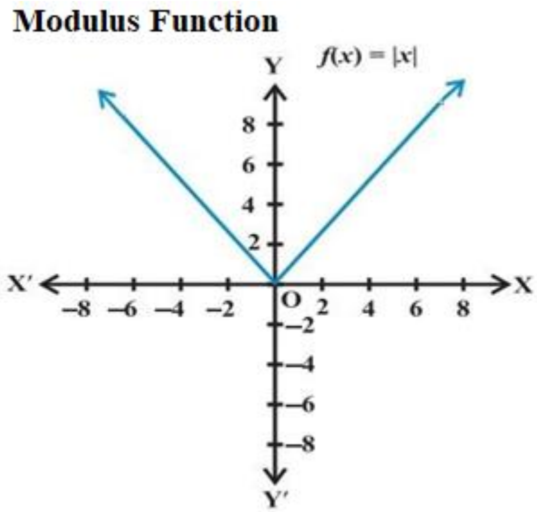
f(*x*) = |*x*| for each *x*∈R,

is called modulus function or absolute value function. For each non-negative value of *x*, *f*(*x*) is equal to *x*. But for negative values of *x*, the value of *f*(*x*) is the negative of the value of *x*, i.e.,

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

The domain of *f* is R and its range is [0,∞)

The graph of the modulus function is;



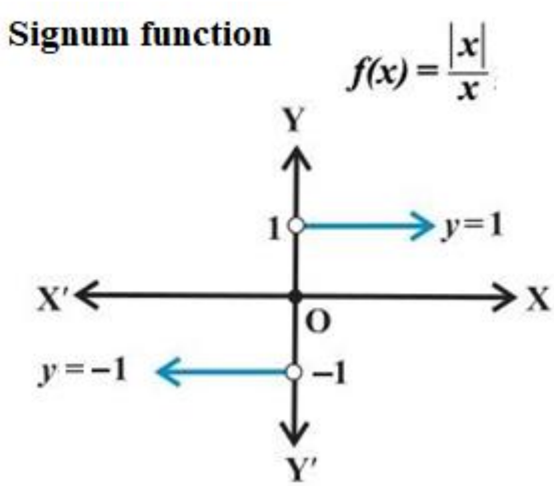
Signum function:

The function f:R→R defined by,

$$f(x) = \begin{cases} 1, & if \;\; x > 0 \\ 0, & if \;\; x = 0 \\ -1, & if \; x < 0 \end{cases}$$

is called the Signum function.The domain of the Signum function is R and the range is the set {–1, 0, 1}.

The graph of the Signum function is given below,



Greatest Integer Function:

The function f: R ⟶ R defined by $f(x) = [x]$, $x∈$R assumes the value of the greatest integer, less than or equal to x. Such a function is called the greatest integer function or floor function or step function.

From the definition of $[x]$, we can see that

$[x] = –3$ for $–3 \leq x < –2$

$[x] = –2$ for $–2 \leq x < –1$

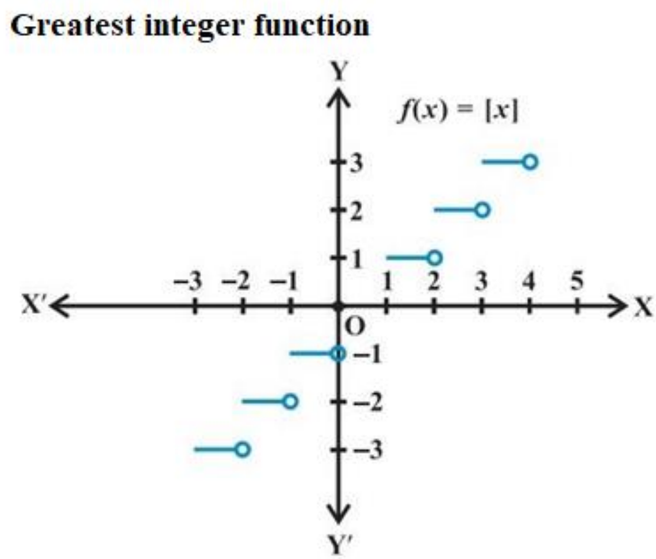
$[x] = –1$ for $–1 \leq x < 0$

$[x] = 0$ for $0 \leq x < 1$

$[x] = 1$ for $1 \leq x < 2$

$[x] = 2$ for $2 \leq x < 3$ and so on.

Its domain is R (set of real numbers) and range is the set of integers. The graph of the function is shown below;



Algebra of Real Functions:

After learning some types of functions, let us now learn how to add two real functions, subtract a real function from another, multiply a real function by a scalar (here by a scalar we mean a real number), multiply two real functions and divide one real function by another.

Addition of Two Real Functions:

Let, $f : X \to R$ and $g : X \to R$ be any two real functions, where $X \subset R$. Then, we define $(f + g): X \to R$ by,

$(f + g)(x) = f(x) + g(x)$, for all $x \in X$.

Subtraction of a Real Function from Another:

Let $f : X \to R$ and $g: X \to R$ be any two real functions, where $X \subset R$. Then, we define $(f - g): X \to R$ by $(f - g)(x) = f(x) - g(x)$, for all $x \in X$.

Multiplication by a scalar:

Let f : X→R be a real valued function and α be a scalar. Here by scalar, we mean a real number.Then the product α f is a function defined as;

(α f): X → Rdefined by(α f) (x) = α f (x), for all x∈X.

Multiplication of two Real Functions:

The product (or multiplication) of two real functions,

f :X→R and g:X→R is a function fg:X→R defined by

(fg) (x) = f(x) g(x), for all x∈X.

This is also called pointwise multiplication.

Quotient of two Real Functions:

Let f : X → R and g: X → R be any two real functions, where X⊂R. Then, the quotient of f by g denoted by$\frac{f}{g}$, is a function,

$\left(\frac{f}{g}\right)$: X→ R defined by$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$, x ∈X

Example 7:

Let $f(x) = x^2$and $g(x) = 2x + 1$ be two real functions. Find

$(f + g)(x)$,$(f - g)(x)$,$(fg)(x)$,$\left(\frac{f}{g}\right)(x)$

Solution:

We have,$(f + g)(x) = f(x) + g(x)$

$\therefore (f + g)(x) = (x^2)+(2x + 1) = x^2+ 2x + 1$

$(f-g)(x) = f(x) - g(x)$

$\therefore (f - g)(x) = (x^2) - (2x + 1) = x^2 - 2x - 1$

$(fg)(x) = f(x)\, g(x)$

$\therefore (fg)(x) = (x^2)(2x + 1) = 2x^3+x^2$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}=\frac{x^2}{2x+1}$ provided $g(x) \neq 0$,$i.e.\ x = -\frac{1}{2}$

Example 8:

Let $f(x) = \frac{1}{x+4}$ and $g(x) = (x + 4)^2$ be two real functions. Find

$(f + g)(x)$, $(f - g)(x)$, $(f g)(x)$, $\left(\frac{f}{g}\right)(x)$, $2f$

Solution:

We observe that $f(x) = \frac{1}{x+4}$ is defined for all $x \in R$, $x \neq -4$.

$\therefore$ Domain of $f = R - \{-4\}$

And $g(x) = (x + 4)^2$ is defined for all $x \in R$,

$\therefore$ Domain of $f \cap$ Domain of $g = R - \{-4\}$,

Hence, $(f + g): R - \{-4\} \to R$, is given by

$(f + g)(x) = f(x) + g(x)$

$\therefore (f + g)(x) = \frac{1}{x+4} + (x + 4)^2 = \frac{1+(x+4)^3}{x+4}$

Similarly, $(f - g): R - \{-4\} \to R$, is given by

$(f - g)(x) = f(x) - g(x)$

$\therefore (f - g)(x) = \frac{1}{x+4} - (x + 4)^2 = \frac{1-(x+4)^3}{x+4}$

Now, $(fg): R - \{-4\} \to R$, is given by

$(fg)(x) = f(x) \times g(x)$

$\therefore (fg)(x) = \frac{1}{x+4} \times (x + 4)^2 = (x + 4)$

Since, $g(x) \neq 0$ for all $x \in R$, except $x = -4$

Therefore, $\left(\frac{f}{g}\right): R - \{-4\} \to R$, is given by

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$= \frac{1}{(x+4)^3}$

5. Summary:

1) A function f from a set A to a set B is a special type of relation for which every element x of set A has one and only one image y in set B.
2) We write f: A→B, where $f(x) = y$.
3) A is the domain and B is the codomain of function f.
4) The range of the function is the set of images.
5) A real function has the set of real numbers or one of its subsets both as its domain and as its range.
6) Algebra of functions:

   For functions $f : X \to R$ and $g : X \to R$, we have

   $(f + g)(x) = f(x) + g(x)$, for all $x \in X$

   $(f - g)(x) = f(x) - g(x)$, for all $x \in X$

   $(fg)(x) = f(x)\, g(x)$, for all $x \in X$

   $(\alpha f)(x) = \alpha f(x)$, for all $x \in X$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$, $x \in X$